Poset Lattice

Solutions

By Abhishek Navlakhi

Semester 3: Discrete Structures

Posets & Lattices

Q.1 (a) Let $Q \in A$

$\because Q \subseteq Q$ $\therefore$ Reflexive

If $P \in A$ & $P \subseteq Q$

$Q \not\subseteq P$ (if $P \neq Q$)

$\therefore \subseteq$ is antisymmetric

Let $R \in A$ such that

$P \subseteq Q$ & $Q \subseteq R$

$\therefore P \subseteq R$

$\therefore \subseteq$ is transitive

$\therefore \subseteq$ is a partial order.

(b) Let $a \in Z^+$

$\therefore a \leq a$ $\therefore$ Reflexive

Let $b \in Z^+$ such that

$a \leq b \Rightarrow b \not\leq a$ (if $a \neq b$)

$\therefore \leq$ is antisymmetric

Let $c \in Z^+$ such that

$a \leq b$ & $b \leq c$

$\therefore a \leq c$

$\therefore \leq$ is transitive

$\therefore \leq$ is a partial order.

(c) Let $a \in Z^+$

$a | a$ is true $\therefore$ Reflexive

Let $b \in Z^+$ such that

$b | a$ (b divides a)

$a \nmid b$ (if $a \neq b$)

$\therefore$ Antisymmetric

Let $c \in Z^+$ such that

b/a and c/b

$\therefore c/a$ (c divides a)

$\therefore$ Transitive

$\therefore$ divisibility is a partial order.

(d) Let $a \in Z^+$

$\therefore a \nless a$ $\therefore$ Not Reflexive

$\therefore$ Not a partial order.

Ex2. [CANCELLED]

Ex3. S^n are all words of length 'n'. For lexicographic ordering we check the first letter. If they differ then the words are arranged based on that letter. But if the letters are same then we check the second & so on.

Basically the ordering is done on the basis of first mismatching character.
S^* is a set of words of 0 or more characters.
First the null string is taken, then the strings are arranged based on the first mismatching character of the words.
If one of the ~~two~~ words ends before getting the first mismatching character then the smaller word is placed before the longer word.

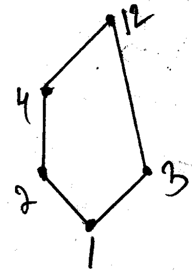
P.4 (a) $\because$ $15 \not< 12$
$\therefore$ False

(b) $\because$ $4 = 4$
& $8 \not< 6$
$\therefore$ False

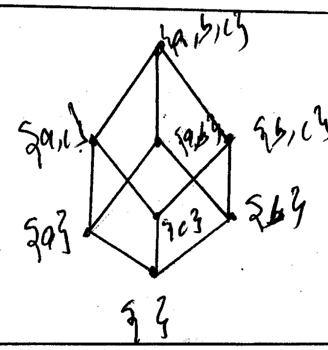
(c) $\because$ $3 = 3$
& $6 < 24$
$\therefore$ True.

(d) $2 < 5$
TRUE.

Ex 4.

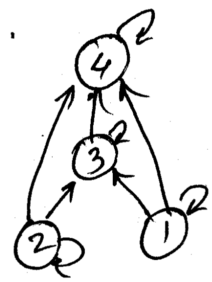


Ex 5.



No.

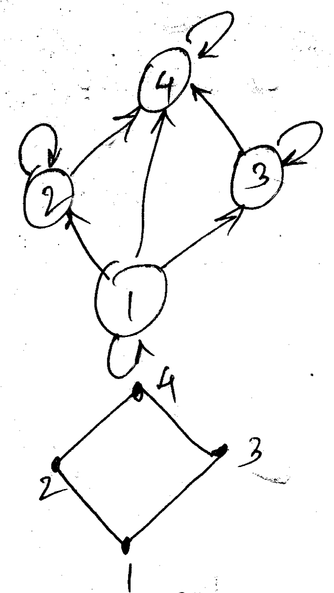
Ex 6.



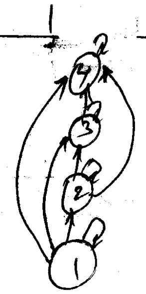
$R = \{(1,1),(1,3),(1,4),$
$(2,2),(2,3),(2,4),$
$(3,3),(3,4),$
$(4,4)\}$

2

P.6

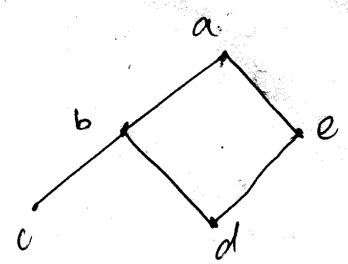


P.8

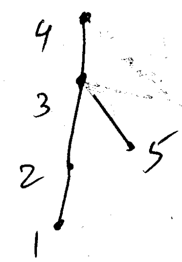


R = {(1,1), (1,2), (1,3), (1,4),
(2,2), (2,3), (2,4),
(3,3), (3,4),
(4,4)
}

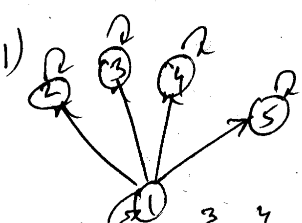
(9)



P.10

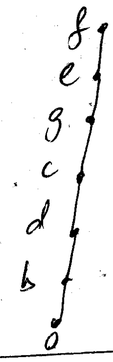


P.12 (i)



$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Ex7



Ex8 If R is a linear order then any/every element x & y are comparable. Say $(x, y) \in R$ then $(y, x) \in R^{-1}$

$\therefore$ Yet x & y are going to be comparable in R^{-1}.

$\therefore$ R^{-1} is also a linear order.

Ex9 let $U \in A$

$U \subset U$

$\therefore$ Irreflexive

let $T, W \in A$ such that

$U \subset T$ & $T \subset W$

$\therefore U \subset W$

$\therefore$ Transitive

$\therefore \subset$ is a quasi-order.

Ex12 Maximal = $\{a, b\}$

Minimal = $\{d, e\}$

Ex13 Maximal = ~~not defined~~ $\{\}$

Minimal = $\{0\}$

P19 (a) Maximal = $\{3, 5\}$

Minimal = $\{1, 6\}$

(b) Maximal = $\{e, f\}$

Minimal = $\{a\}$

(c) Maximal = $\{f, g\}$

Minimal = $\{a, b, c\}$

(d) Maximal = $\{4, 7\}$

Minimal = $\{1, 9, 8\}$

P.20 (a) Maximal = ~~undefined~~ = $\{\}$
Minimal = ~~undefined~~ = $\{\}$
(b) Maximal = ~~undefined~~ = $\{\}$
Minimal = $\{0\}$
(c) Maximal = $\{1\}$
Minimal = ~~undefined~~ $\{\}$
(d) Maximal = $\{4, 8\}$
Minimal = $\{2, 3\}$

Ex 14. (a) Greatest = not defined.
Least = $\{0\}$
(b) [Assuming relation $\subseteq$]
Greatest = $\{a, b, c\}$
Least = ϕ
(d) Greatest = 1
Least = 0
(c) Greatest = not defined
Least = not defined.
(e) Greatest = not defined
Least = Not defined.

Ex 15. (i) Greatest = f
Least = a
(ii) Greatest = e
Least = not defined.

P.21 (a) Greatest = not defined
Least = not defined
(b) Greatest = 5
Least = not defined
(c) Greatest = not defined
Least = not defined
(d) Greatest = 72
Least = 2

Ex 16 (i) UB = $\{c, d, e, f, g, h\}$
LB = $\{\}$
(ii) UB = $\{f, g, h\}$
LB = $\{c, a, b\}$

Ex 17. (i) LUB = c
GLB = not defined.
(ii) LUB = not defined
GLB = c.

Ex 18 LUB = 10
GLB = 4.

Q22 (i) UB = {f, g, h}

LUB = f

LB = {c, a, b}

GLB = c

(ii) UB = { }

LUB = not defined

LB = { }

GLB = not defined

(iii) UB = {d, e, f}

LUB = d

LB = {b, a}

GLB = b

(iv) Taking B = {4, 6, 7}

UB = {5}

LUB = 5

LB = { }

GLB = not defined.

Ex 19. For any A, B ∈ L

LUB = A ∪ B

GLB = A ∩ B

∴ is a lattice.

Ex 20. For a, b ∈ Z^+

LUB = LCM(a, b)

GLB = GCD(a, b)

Hence $(Z^+, \leq)$ is a lattice

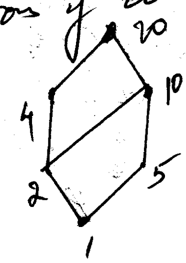
Ex 21. Yes if a, b ∈ Dn

GLB = GCD(a, b)

LUB = LCM(a, b)

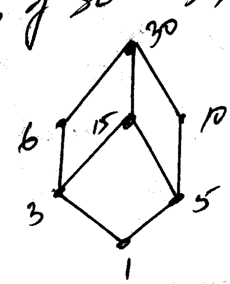
D20

Factors of 20 ⇒ 20, 10, 5, 4, 2, 1



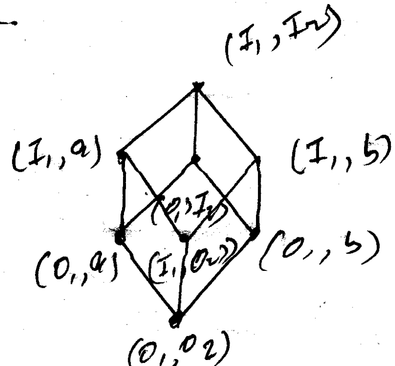
D30

Factors of 30 ⇒ 30, 15, 10, 6, 5, 3, 1



Ex 22. (i) Yes
(ii) Yes
(iii) No ∵ LUB(f, g) is not defined.
(iv) Yes
(v) Yes
(vi) No LUB(b, c) not defined.
(vii) No GLB(c, d) not defined.

Ex 23



Ex 24. No ∵ GLB(a, b) not defined
No ∵ ~~GLB~~ LUB(a, b) = I & original LUB(a, b) = c
Yes.

Ex 26. No ∵ Z is an infinite set
Greatest = undefined
Least = undefined.

Ex 27. Yes
Greatest = 1 = S
Least = 0 = { }

Ex 28. Yes ∵ every element if it has then has only one complement

~~eg: Element Complement~~

Ex 29. Yes. Since every element either has no complement or has 1 complement

Element	Complement
0	1
1	0
a	—
d	—
b	c
c	b

Ex 30 (a)

Element	Complement
O	I
I	O
a	c
b	c
c	a, b

Since there is atleast one node with 2 complements
∴ not distributive.

$a \wedge (b \vee c) = a \wedge I = a$

$(a \wedge b) \vee (a \wedge c) = b \vee O = b$

∴ $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

∴ Not distributive.

(b)

Element	Complement
a	b, c
b	a, c
c	a, b
O	I
I	O

Since there is atleast one element with more than one complement
∴ Not distributive.

$a \wedge (b \vee c) = a \wedge (I)$
$= a$

$(a \wedge b) \vee (a \wedge c) = O \vee O$
$= O$

∴ $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

∴ Not distributive.

Ex 31. Done in the previous example.

Ex 32. Since every element has a complement
∴ complemented Lattice
If A is the element of P(S) then its complement is S − A

Ex 33. Yes since every element has a complement as shown in Ex 30.