Chapter 3:

Relations & Digraphs

Cartesian Product of sets or Product Sets:

Ordered Pair

An <u>ordered pair</u> (x ,y) is a listing of objects x & y in the specified order, with x appearing first & y appearing second. Thus an ordered pair is a sequence of length 2.

Equality of Ordered pairs

Two ordered pair (x_1 , y_1) & (x_2 , y_2) are equal if & only if $x_1 = x_2$ & $y_1 = y_2$.

Product Set or Cartesian Product

If A & B are two non – empty sets then the <u>product set or Cartesian product</u> AXB is defined as a set of all ordered pairs (x , y) with x∈A & y∈B.
A X B = {(x , y) | x ∈ A and y ∈ B}

Example 1:

Let A = {1, 2, 3, 4} and B = {a, b, c}; find AXB and BXA.

Note: The number of elements (ordered pairs) in AXB & BXA is 12.

Number of elements in A X B =
Number of elements on A X Number of elements in B

| A X B | = | A | | B | ; | B X A | = | B | | A |

Note: A X B ≠ B X A
| A X B | = | B X A |

Theorem 1:

For any two finite, non – empty sets A & B,
|A X B| = |A| |B|

Proof:

If set A consists m elements & set B consists of n elements, then,
| A | = m | B | = n
Finding A X B involves two task, first selecting the first element of the ordered pair from A & secondly selecting an element from set B. i.e. selecting (x , y) such that x ∈ A & y ∈ B.
Selection of x can be done in m ways (since A has m elements) & selection of y can be done in n ways (since B has n elements). Thus by principle of multiplication we can say that finding elements (x,y) of A X B can be done in m .n ways.
i.e. | A X B | = m . n = | A | | B |

Example 2:

If A = B = R, where R is a set of all real numbers, then, what does (x, y) signify, where x ∈ A and y ∈ B?

Example 3:

A survey is to be conducted by Govt. of India. The survey will categorize all male & female Indians based of their education (elementary school, high school, junior college, graduate school, post graduate school). Design the sets & define all possible combinations.

Cartesian Product of n sets

If $A_1, A_2, A_3, \ldots, A_n$ are n non – empty finite sets then $A_1 X A_2 X A_3 X \ldots X A_n$ is a set of ordered pairs of n – tuples $(x_1, x_2, x_3, \ldots, x_n)$ where $x_1 \in A_1$, $x_2 \in A_2$, $x_3 \in A_3$ ….. & $x_n \in A_n$. Thus,

$A_1 X A_2 X A_3 X \ldots X A_n = \{(x_1, x_2, x_3, \ldots, x_n) | x_i \in A_i, i = 1, 2, 3, \ldots, n\}$

Example 4:

Consider a company that plans releasing it two new software programs for UNIX, WINDOWS98 & WINDOWS XP. As a promotional offer it decided to release a demo version of the software. Help the company in deciding the number of possible combinations it will have to handle in releasing its demo & full versions.

More Examples:

Example 5:

Find the value of x & y so that the following statement is true

(a) (y, 3) = (4, 3)
(b) (a, 3x) = (a, 9)
(c) (3y + 1, 2) = (7, 2)
(d) (C, C++) = (y, x)

Example 6:

If A = {a , b} & B = { 1, 2, 3 }, find

(a) A X B (b) B X A (c) A X A (d) B X B

Example 7:

If A = {a, b}, B = {1, 2, 3} & C = {*, #} find A X B X C

Example 8:

If A = {x | x is a real number} & B = {2, 4, 6} sketch each of the following in the Cartesian plane.

(a) A X B (b) B X A

Example 9:

For sets A, B & C investigate whether

A X (B ∩ C) = (A X B) ∩ (A X C).

Try it yourself

Problem 1:

Find x & y

(a) (4y, 6) = (16,x)
(b) (2y – 3, 3x – 1) = (5, 7)
(c) $(x^2, 36) = (81, y)$
(d) $(x, y) = (x^2, y^2)$

Problem 2:

A car manufacturer decides to launch three cars Accent, Santro, Xing in two possible types of engines – petrol & diesel. Help the company by giving it the count & listing all the possible models & their engine types.

Problem 3:

If $A = \{a \mid a \text{ is a real number } \& -2 \leq a \leq 3\}$

$B = \{b \mid 1 \leq b \leq 5 \wedge b \in R\}$,

sketch the following in the Cartesian plane:

(a) A X B (b) B X A

Problem 4:

Show that if A_1 has n_1 elements, A_2 has n_2 elements & A_3 has n_3 elements, then $A_1 X A_2 X A_3$ has $n_1.n_2.n_3$ elements.

Problem 5:

If $A \subseteq C$ and $B \subseteq D$, prove that $A X B \subseteq C X D$.

Problem 6:

Let A, B & C be subsets of ξ. Prove that AX(BUC)=(AXB)U(AXC)

Partition Set or Quotient Set:

Partition Set or Quotient Set

A partition set or quotient set of 'A' is a collection of sets A_1, A_2, A_3, etc. such that

- each element of set A belong to one of the sets of the partition i.e. A_1, A_2, A_3, etc.
- If A_1 & A_2 are two elements of the partition (subsets of P(A)) then $A_1 \cap A_2 = \emptyset$

Example 10:

Let A = {1,2,3,4,5,6,7,8,9,10}

$A_1 = \{1,2,3,4\}$ $A_2 = \{5,6,7\}$ $A_3 = \{4,5,7,9\}$

$A_4 = \{4,8,10\}$ $A_5 = \{8,9,10\}$

$A_6 = \{1,2,3,6,8,10\}$

Which of the following are partitions of A?

(a) $\{A_1,A_2,A_5\}$ (b) $\{A_1,A_3,A_5\}$ (c) $\{A_3,A_6\}$

(d) $\{A_1,A_2\}$

Example 11: List all partitions of {a,b,c,d}

Example 12: Give the partition of {0,3,6,9,……} containing

(a) two infinite sets
(b) three infinite sets

Try it yourself

Problem 7: List all partitions of {1, 2, 3}

Problem 8: List one partition of Z

Relations:

Relation

Assume that we have a set of men *M* and a set of women *W*, some of whom are married. We want to express which men in *M* are married to which women in *W*. One way to do that is by listing the set of pairs (*m,w*) such that *m* is a man, *w* is a woman, and *m* is married to *w*. So, the relation "married to" can be represented by a subset of the Cartesian product $M \times W$. In general, a *relation* R from a set *A* to a set *B* will be understood as a subset of the Cartesian product $A \times B$, i.e., $R \subseteq A \times B$. If an element $a \in A$ is related to an element $b \in B$, we often write aRb instead of $(a, b) \in R$.

The set

$\{a \in A \mid aRb \text{ for some } b \in B\}$ is called the *domain* of R.

The set

$\{b \in B \mid aRb \text{ for some } a \in A\}$ is called the *range* of R.

For instance, in the relation "married to" above, the domain is the set of married men, and the range is the set of married women.

If *A* and *B* are the same set, then any subset of $A \times A$ will be a *binary relation* in *A*. For instance, assume $A = \{1, 2, 3, 4\}$. Then the binary relation "less than" in *A* will be:

$R = \{(x, y) \in A \times A \mid x < y\}$

$= \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

Notation: A set *A* with a binary relation R is sometimes represented by the pair (*A*,R). So, for instance, (Z,<) means the set of integers together with the relation of strict inequality.

Definition

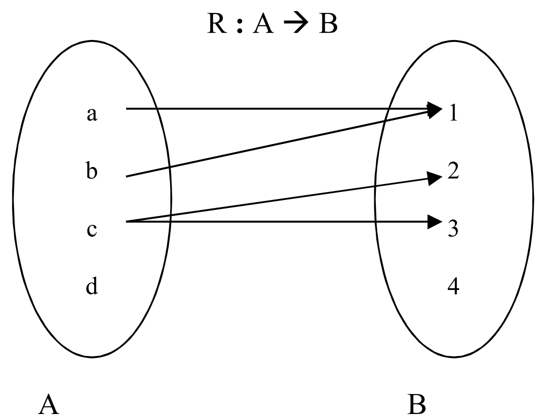
If A & B are two non – empty sets then R is said to be a relation from A to B if R is a subset of A X B.

If (a,b) є R, we say that a **is related to** b **by R**, which can be written as a R b. If a is not related to b by R, we write as $a \not R b$.

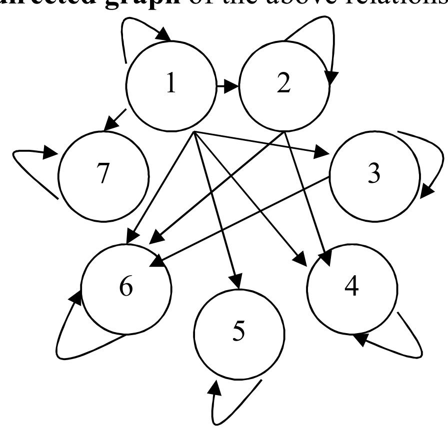
Representation

Arrow Diagram or digraph

Venn diagrams and arrows can be used for representing relations between given sets.
A = *{a, b, c, d}* to *B* = *{*1, 2, 3, 4*}* given by
R = *{*(*a,* 1)*,* (*b,* 1)*,* (*c,* 2)*,* (*c,* 3)*}*.
In the diagram an arrow from *x* to *y* means that *x* is related to *y*.



Let as find the factor relationship ('is a factor of ') on
A = {1, 2, 3, 4, 5, 6, 7}
The relation is
R= {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,2),(2,4),(2,6),
(3,3),(3,6),(4,4),(5,5),(6,6),(7,7)}
The **diagraph or directed graph** of the above relationship is as follows:



The circles are called **vertices**. The arrow is called an **edge**. An edge exists between two vertices a & b if & only if aRb. Thus the edges correspond to the ordered pairs in R, & the vertices correspond exactly to the elements of set A.

Matrix representation

Another way of representing a relation R from *A* to *B* is with a matrix. Its rows are labeled with the elements of *A*, and its columns are labeled with the elements of *B*. If *a є A* and *b є B* then we write 1 in row *a* column *b* if *a*R*b*, otherwise we write 0.
For instance the relation R = *{*(*a,* 1)*,* (*b,* 1)*,* (*c,* 2)*,* (*c,* 3)*}* from *A* = *{a, b, c, d}* to *B* = *{*1, 2, 3, 4*}* has the following matrix:

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} 1\ 2\ 3\ 4 \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Example 13: Let A = {1,2,3,4}. Define the following relation R (is less than) on A: aRb if & only if a < b

Example 14: If A=R, the set of real numbers, then plot the relation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Example 15: Consider the following tariff plan of ABC airlines between 4 cities of India p, q, r, & s.

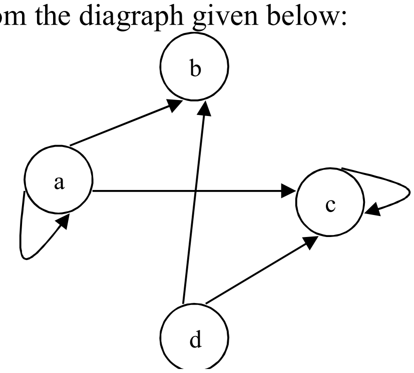
Tariff in Rs.	Destination			
	P	q	r	s
p		2500	2000	1350
q	1500		3500	2000
r	1000	3650		3200
s	1200	2500	1700	

Now the company decides to launch new off – season offers. Help the company in deciding the journey along which the offers can be applied if the decision was taken to give offers only on journeys costing Rs.1750 & more.

Note: The table represents a relation giving the rates of the journey between p, q, r & s. e.g. p to q is Rs.2500 & q to p is Rs.1500 & so on…

Example 16: Let A= {1,2,3,4} &
R = {(1,1),(1,2),(2,1),(2,2),(2,3),(2,4),(3,4),(4,1)}
Represent the diagraph for the above relationship.

Example 17: Find the relation from the diagraph given below:



In – degree & out– degree

In – degree of a vertex is the number of edges terminating at that vertex.
Out – degree of a vertex is the number of edges leaving the vertex.
e.g. the in – degree of vertex a in example 17 is 1 & the out – degree of vertex a is 3.

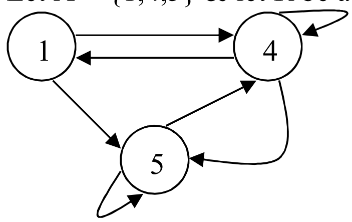
Example 18: Let A={a,b,c,d} & let R be the relation on A such that

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Draw the diagraph & find the in – degree & out – degree of each vertex.

Hint: The out – degree of node a is the number of edges leaving vertex a i.e. number of 1's in row a of the matrix
The in – degree of node a is the number of edges ending in a i.e. number of 1's in column a of the matrix.

Example 19: Let A = {1,4,5} & let R be a relation on A given by:



Find R & M_R

Example 20: Find the domain, range & the matrix relation of R if
R: A → B where A={a,b,c,d} & B={1,2,3}
R={(a,1),(a,3),(b,1),(b,2),(c,2),(c,3)}

Example 21: If A = {1,4,6,8,9} & B={1,2,3}
& R: A → B such that aRb if & only if $a = b^2$
Find the domain, range, co – domain & the matrix relation.

Example 22: If A = {1,2,3,4,6,8} = B & aRb if & only if a is a multiple of b & R: A→B.
Find M_R, domain, range, in–degree, out-degree & draw a digraph.

Note: In – degree can be got by counting the number of 1's in the column of the above matrix for that corresponding vertex.
Out – degree can be got by counting the number of 1's in the row of the above matrix for that corresponding vertex.
Alternately, the in – degree can be got from the digraph by counting the number of edges terminating on that vertex & out – degree can be got from the diagraph by counting the numebr of edges leaving that vertex.

Try it yourself

Problem 9: Find the domain, range, R, M_R, in – degree & out – degree of each vertex & also draw the digraph, if
A = {2,4,6,8}=B & R: A → B if & only if a = b

Problem 10: A = {1,2,5,7,9}=B
R: A → B
aRb if & only if $a \leq b$.
Find the domain, range, R, M_R, diagraph, in – degree & out – degree of each vertex.

Problem 11: If A = {1,3,5} , B = { 2,4,6,8} & aRb if & only if $b < a$ & R: A → B
Find the domain, range, R & M_R.

More Examples:

Example 23: If $A = Z^+$, the set of positive integers, & R is defined as aRb if & only if there exists a k in Z such that $a=b^k$, find which of the following belong to R?

(a) (4,16) (b) (2,32) (c) (3,3)
(d) (2,8) (e) (8,2) (f) (1,7)

Example 24: If A = R (the set of real numbers) & if G is a relation on A such that aGb if & only if $a^2 + b^2 = 25$, find domain & range.

Example 25: Let A be the product set {1,2,4} X {c,d}. How many relations are there on A?

Try it yourself

Problem 12: If A = R, the set of real numbers & G is a relation on A such that aRb if and only if $2a + 3b = 6$. Find Dom(G) & Ran(G).

Problem 13: Find R & draw a digraph if
A = {1,2,3,4}
&

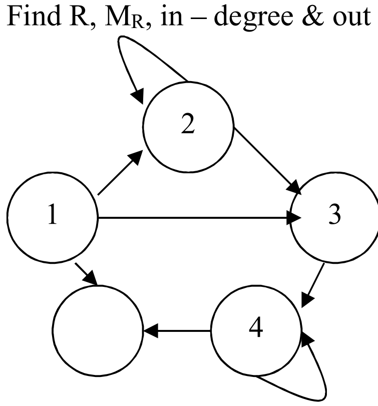
$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Problem 14: If A = {a,b,c,d}

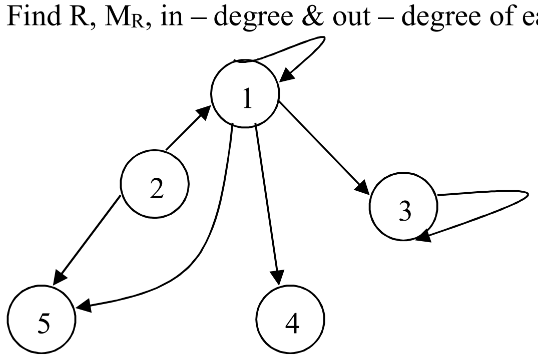
$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Find R & draw a diagraph of R

Problem 15: Find R, M_R, in – degree & out – degree of each vertex



Problem 16: Find R, M_R, in – degree & out – degree of each vertex.



Restriction of a Relation

If R is a relation on set A i.e. $R\subseteq A \times A$ & B is a subset of A (i.e. $B\subseteq A$) then the **restriction of R to B** is
$R \cap (B \times B)$.

Example 26:

If A={a,b,c,d,e}
R = {(a,a),(a,c),(b,c),(a,e),(b,e),(c,e)}
& B ={a,b,c}
Find the restriction of R to B

Try it yourself Problem 17:

If A={(1,1),(1,3),(1,5),(2,2),(2,4),(3,2),(3,5),(4,1),(5,1),(5,3)}
Compute the restrictio of R to B for the subset of A given by
(a) B={1,3,4,5}
(b) B={2,3,4}

R – relative set

If R is a relation from A to B R: A→B then R – relative set of x, where x ϵ A, are those values of y ϵ B for which x is R-related to y. (i.e. xRy)
R(x) = {yϵB | xRy}

Note: If $A_1\subseteq A$ & R: A→B then R-relative set of A_1 is
$R(A_1)$={yϵB | xRy for some x in A_1}

Example 27:

If A={a,b,c}
R={(a,a),(a,c),(b,a),(b,c),(c,a)}
Find R(a), R(b), R(c).
If A_1={b,c} find $R(A_1)$

Theorem 2:

(a) If $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$

Proof:
Let x ϵ A_1 & y ϵ $R(A_1)$ (i.e. xRy is true)
Since $A_1\subseteq A_2$ & x ϵ A_1
∴ x ϵ A_2 & we know that xRy
∴ y ϵ $R(A_2)$
∴ All elements of $R(A_1)$ belongs to $R(A_2)$
But there could be an element like q ϵ A_2 & q∉ A_1. Thus all elements of $R(A_2)$ may not be a part of $R(A_1)$.
Thus $R(A_1) \subseteq R(A_2)$ if $A_1\subseteq A_2$

(b) If $A_1\subseteq A_2$ then $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

Proof:
Let xϵ$A_1 \cup A_2$ &
yϵ$R(A_1 \cup A_2)$ (i.e. xRy is true) …………(I)
∴ xϵA_1 &/or xϵA_2
& we know that xRy
∴ yϵ$R(A_1)$ &/or yϵ$R(A_2)$
∴ yϵ$R(A_1) \cup R(A_2)$ ……………………..(II)
From I & II we can say that
$R(A_1 \cup A_2) \subseteq R(A_1) \cup R(A_2)$ ……………(III)
Since A1 $\subseteq$A2
∴ $A_1 \subseteq A_1 \cup A_2$
∴ $R(A_1) \subseteq R(A_1 \cup A_2)$ …………………(IV)
Similarly $A_2\subseteq A_1 \cup A_2$ (in fact $A_2 = A_1 \cup A_2$)
∴ $R(A_2)\subseteq R(A_1 \cup A_2)$ …………………..(V)
From IV & V

$R(A_1) \cup R(A_2) \subseteq R(A_1 \cup A_2)$ …………(VI)
From III & VI we can say that
$R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

(c) If $A_1 \subseteq A_2$ then $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

Proof: Let $x \in A_1 \cap A_2$ &
$y \in R(A_1 \cap A_2)$ (i.e. xRy) ……………….(I)
$\therefore x \in A1$ & $x \in A_2$
Since xRy
$\therefore y \in R(A_1)$ & $y \in R(A_2)$
$\therefore y \in R(A_1) \cap R(A_2)$ ……………………….(II)
From I & II
$R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

Example 28: If A=Z, set of integers, & let R be the relation $\leq$ on A. Let A1={0,1,2} & A2={7,8}
Find $R(A_1)$, $R(A_2)$, $R(A_1 \cup A_2)$, $R(A_1 \cap A_2)$

Example 29: If A={1,2,3,4} B={a,b,c,d}
& R: A→ B is given by
R= {(1,a),(1,c),(2,a),(2,b),(2,d),(3,b),(3,c),(4,c),(4,d)}
Let A_1={1,2}& A_2={2,3} find $R(A_1 \cap A_2)$

Try it yourself

Problem 18: If A = {1,2,3,4} and B = {1,4,6,8}
& aRb if & only if a divides b & R: A→B
Find $R(A_1)$ if (a) A_1= {1,3} (b) A_1={1,3,4}

Problem 19: If A={1,2,4,6}=B and R:A→B and aRb if and only if a is a multiple of b. Find
(a) R(2) (b) R(4) R{1,4,6}

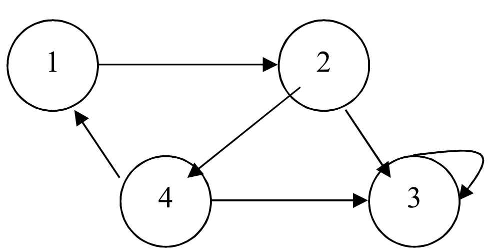
Problem 20: If R: A→B & A_1 & A_2 are subsets of A, then prove that
$R(A_1 \cap A_2) = R(A_1) \cap R(A_2)$ if & only if $R(a) \cap R(b)$={ } where a & b are distinct elements of A.

Example 30: If A has m elements & B has n elements, how many different relations are possible from A to B?

PATHS

Path If R is a relation on A, then a path of length n from a to b is a sequence of n edges starting from a & ending at b, where the vertices (traversed in the direction marked on the edges) need not be all distinct.
A path is written as
$\pi : a, x_1, x_2, x_3, \ldots\ldots , x_{n-1}, b$
where aRx_1, x_1Rx_2, x_2Rx_3,……………, $x_{n-1}Rb$

Example 31:



Give a path & its length from vertex (i) 1 to 3 (ii) 2 to 2 (iii) 3 to 3

Cycle

A cycle is a path that starts & ends at the same vertex.

Connectivity Relation

If there exists a path of length n from x to y where x,y ϵ A & R is a relation on A, then we can symbolize this as $xR^n y$.
$xR^\infty y$ means that there exists a path in R from x to y of some length (≥1). This relationship (R^∞) is called connectivity relation of R.

Note: $R^n(x)$ consists of a set of all yϵA such that there exists the relationship $xR^n y$, i.e. $R^n(x)$ is a set of all vertices of R such that there exists a path of length n from x to that vertex.
Similarly $R^\infty(x)$ consists of a set of all vertices which can be reached from x (of any path length).
e.g. say a passenger wants to take a flight from India to Kuwait, then if Kuwait ϵ R^∞(India) then it is possible to go to Kuwait from India.
If Kuwait ϵ R^2(India), then the person can do to Kuwait via an intermediate stop (path length =2).
In general, if Kuwait ϵ R^n(India) then the person can go to Kuwait via n –1 intermediate stops.

Example 32:

If A={a,b,c,d,e,f}
& a relation R on A is given by
R={(a,b),(a,c),(b,b),(b,d),(b,e),(c,d),(d,e),(e,f)}
Draw the digraph of the relation R^2 on A & also find R^∞.

Try it yourself
Problem 21:

If A={1,2,3,4,5} & R is a relation on A & is given by
R={(1,1),(1,2),(2,3),(3,5),(3,4),(4,5)}
Find (i) R^2 (ii) R^∞

Boolean Matrix Multiplication
Finding M_R^n

$$M_R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_R^2 = M_R \odot M_R = (M_R)^2_{\odot}$$

$$= \begin{pmatrix} 1\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1 \\ 0\,0\,1\,1\,0 \\ 1\,0\,0\,0\,0 \end{pmatrix} \odot \begin{pmatrix} 1\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1 \\ 0\,0\,1\,1\,0 \\ 1\,0\,0\,0\,0 \end{pmatrix}$$

$$= \begin{pmatrix} 1.1+1.0+0.0+0.0+0.1\;\; 1 & 1 & 0 & 0 \\ 0.1+0.0+1.0+0.0+0.1\;\; 0 & 0 & 0 & 1 \\ 0.1+0.0+0.0+0.0+1.1\;\; 0 & 0 & 0 & 0 \\ 0.1+0.0+1.0+1.0+0.1\;\; 0 & 1 & 1 & 1 \\ 1.1+0.0+0.0+0.0+1.1\;\; 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Note: $1.1=1$ $1.0=0.1=0$ $1+0=0+1=1$ $0+0=0$

Note: M_R^2 can also be constructed from R^2. But finding R^2 from M_R^2 will be more convenient.
Generalizing the above fact we can say
$M_R^n = M_R \odot M_R \odot M_R \odot \ldots\ldots \odot M_R$ (n terms)

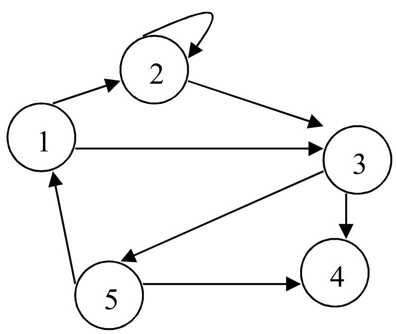
Finding M_R^∞

If xRy means there is a path of length 1 from x to y.
Similarly if xR^2y means there is a path ofc length 2 from x to y.
If $xR^\infty y$ means there is a path of any length (≥1) from x to y. i.e. y can be reached from x irrespective of the number of intermediate vertices.

$\therefore R^\infty = R \cup R^2 \cup R^3 \ldots\ldots\ldots$

$\& \; M_R^\infty = M_R \vee M_R^2 \vee M_R^3 \ldots\ldots\ldots$
$= M_R \vee (M_R \odot M_R) \vee (M_R \odot M_R \odot M_R) \ldots\ldots\ldots$
$= M_R \vee (M_R)^2_{\odot} \vee (M_R)^3_{\odot} \vee \ldots\ldots\ldots$

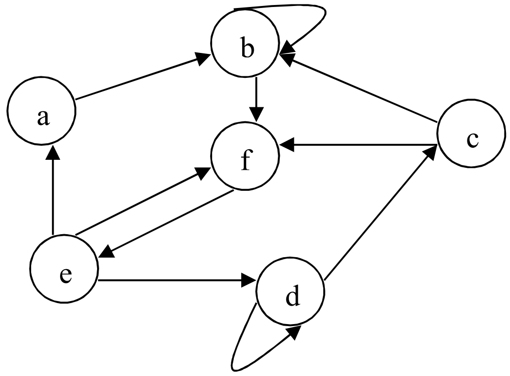
Example 33: Consider the following digraph



Find:
(1) All paths of length 1
(2) All paths of length 2
(3) All paths of length 2 starting at vertex 2.
(4) All paths of length 3
(5) All paths of length 3 from vertex 3
(6) a cycle starting at vertex 1
(7) Draw the digraph of R^2.
(8) MR^2
(9) R^∞
(10) M_R^∞

Try it yourself

Problem 22:



Find

(1) All paths of length 1
(2) All paths of length 2
(3) All paths of length 2 from vertex e.
(4) All paths of length 3
(5) All paths of length 3 from vertex e
(6) A cycle starting at e
(7) Draw the digraph of R^2
(8) $M_R{}^2$
(9) R^∞
(10) $M_R{}^\infty$

Problem 23: Let R & S be relations on set A. Show that

$M_{RUS} = M_R \vee M_S$

Problem 24: Let R be a relation on A. Show that

$M_R{}^* = M_R{}^\infty \vee I_n$

where I_n is a n X n identity matrix & n is the number of elements of set A.

Reachability relation (R)*

Reachability between two vertices x & y (i.e. xR^*y) is defined if it is possible to reach y from x.

i.e. if x = y or if $xR^\infty y$

$M_R{}^* = M_R{}^\infty \vee I_n$

where I_n is a n X n identity matrix

$\therefore M_R{}^\infty = [M_R \vee (M_R)^2_\odot \; (M_R)^3_\odot \vee \vee \ldots\ldots] \vee I_n$

If π_1: $a_1, x_1, x_2, x_3, \ldots\ldots\ldots\ldots, x_{n-1}, b$

is a path of n edges from a to b.

& π_2: $b, y_1, y_2, y_3, \ldots\ldots\ldots, y_{m-1}, c$

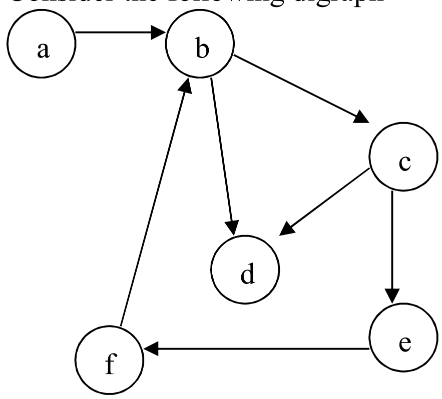
is a path of length m from a to c.

Then it is obvious that path from a to c exists & is given by the composition

$\pi_2 \circ \pi_1$: $a, x_1, x_2, \ldots\ldots, x_{n-1}, b, y_1, y_2, \ldots\ldots, y_{n-1}, c$

which is of length n + m.

Example 34: Consider the following digraph



Find a path from a to c & from c to d of lenth >1 & also find their composition.

Try it yourself

Problem 25: If π_1: 1,3,5,2 π_2: 2,5,7,9
Find π_2 o π_1

Problem 26: If π_1: a,b,d π_2: d,c,f,g
Find the composition π_2 o π_1

PROPERTIES OF RELATIONS:

Types of relations

A relation can be
- Reflexive
- Irreflexive
- Symmetric
- Asymmetric
- Antisymmetric
- Transitive
- Equivalence relation

Reflexive & Irreflexive Relations

A relation is **reflexive** if $(a,a) \in R$ for **all** $a \in A$ i.e. aRa for **all** $a \in A$.
Irreflexive relation is one in which $(a,a) \notin R$ for **all** $a \in A$ i.e. $a \not R a$ for **all** $a \in A$

Note: A realtion may not be reflecxive & neither irrreflexive
The **matrix of a reflexive relation** will have all th emajor diagonal elements as 1.

$$M_R = \begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix}$$

An **irreflexive relation** in a **matrix** form would have all the major diagonal elements as 0.

$$M_R = \begin{bmatrix} 0 & x & x \\ x & 0 & x \\ x & x & 0 \end{bmatrix}$$

In the **digraph** of a **reflexive relation** all the vertices will have cycles of length 1 (i.e. an edge from a vertex to that vertex itself).
The **digraph** of a **irreflexive relation** will have no cycles of length 1.
In a reflexive relation
Dom(R) = Ran(R) = A (where R is a erlation on A)

Symmetric, Asymmetric & Antisymmetric

A relation is a **symmetric** relation if (a,b)ϵR then (b,a)ϵR. i.e if aRb then bRa.
A relation is **asymmetric** if (a,b)ϵR then (b,a) $\notin$ R i.e. if aRb then $b \not R a$
A relation is **antisymmetric** if (a,b)ϵR then (b,a)ϵR only if a=b. i.e. aRb then bRa if & only if a=b.
∴ A relation is **not antisymmetric** if there is some $a \neq b$ & (a,b)ϵR & (b,a)ϵR i.e. There is some aRb & bRa & a≠b.
OR simply if a≠b then $a \not R b$ or $b \not R a$ then R is **antisymmetric.** i.e. if a≠b & (a,b)ϵR then (b,a) $\notin$ R.

Matrix of a **symmetric** relation will be such that
If $m_{ij} = 1$ then $m_{ji} = 1$
& if $m_{ij} = 0$ then $m_{ji} = 0$
i.e. $M_R = M_R^T$

Matrix of a **asymmetric** relation is such that
If $m_{ij} = 1$ then $m_{ji} = 0$
Note: $m_{ii} = 0$ i.e. the main diagonal elements will be zero.

Matrix of a **antisymmetric** relation will be such that
$m_{ij} = 0$ OR $m_{ji} = 0$ for $j \neq i$ (i.e. for non – diagonal elements).
Note: The main diagonal elements may be 1 or 0.

Example 35: Identify which of the following matrix of relations are symmetric, asymmetric, antisymmetric.

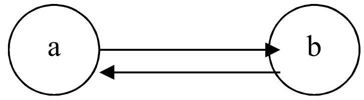
(a)
$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)
$$M_R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

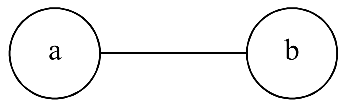
(c)
$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Digraphs

Digraph of a **symmetric relation** is such that if there is an edge from vertex a to vertex b then there should be an edge from vertex b to vertex a also.

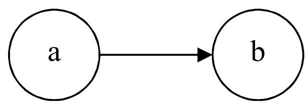


This type of edges from a to b & b to a can be replaced by an unidirectional edge. The resulting figure is called a **graph**. Thus is a graph all the edges are bidirectional i.e. all graphs are symmetric relations.

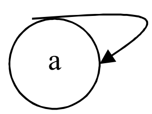


Note: Cycles of length 1 are allowed

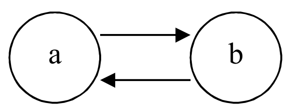
Digraph of a **asymmetric relation** is such that if there is a edge from vertex a to vertex b then there should be no edge from vertex b to vertex a. Even cycles of length 1 are **not** allowed.



Asymmetric

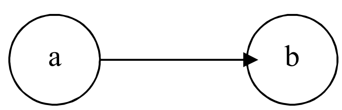


Not asymmetric

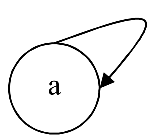


Not asymmetric

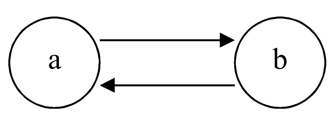
Digraph of a **antisymmteric relation** is such that if a & b are two distinct vertices & an edge exists from a to b the there should be no edge from b to a.
Cycles of length 1 are allowed.



Antisymmetric



Antisymmetric



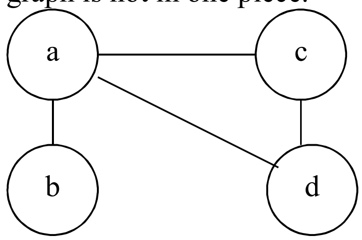
Not Antisymmetric

Example 36:

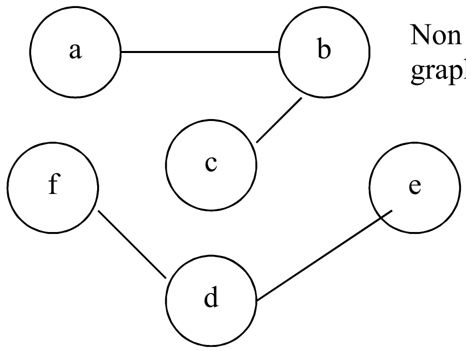
Identify if
R={(a,b),(b,a),(a,c),(c,a),(b,c),(c,b),(b,e),(e,b),(e,d),(d,e), (c,d),(d,c)}
is a symmetric, asymmetric, antisymmteric relation if R is a relation on A where A={a,b,c,d,e}

Connected & Non – connected graphs

A connected graph is one in which there is a path from any one element to any other.
A non – connected graph is one in which it is not possible to go from one vertex to another.
Thus a connected graph is a graph in one piece whereas a non – connected graph is not in one piece.



A connected graph



Non – connected graph

Transitive Relation

A relation is **transitive** if aRb & bRc then aRc
OR a relation is not transitive if aRb & bRc then a$\not R$c
A relation is **transitive** is (a,b)ϵR & (b,c)ϵR then (a,c)ϵR.
In terms of a **matrix** of a relation, a relation is transitive
if $m_{ij} = 1$ & $m_{jk} = 1$ then $m_{ik} = 1$
i.e. there is a 1 in the row of vertex i & the column of vertex k in M_R^2.
There will also be a 1 in m_{ij}, m_{jk}, m_{ik} of M_R.
Thus if $M_R^2 = M_R$ then R is transitive but if R is transitive does not imply that $M_R^2 = M_R$.

Note: If R is transitive then $R^n \subseteq R$

Example 37:

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

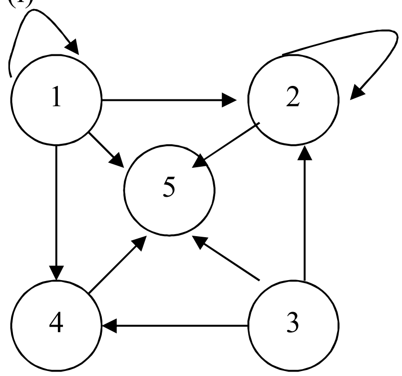
where R is a relation on set A where A={1,2,3}
Show that R is transitive.

Try it yourself
Problem 27:

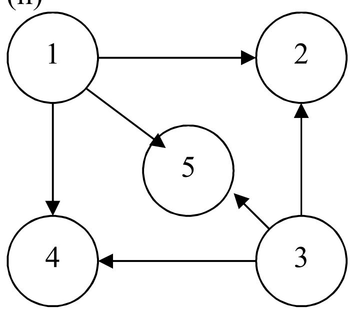
Determine if the relation is reflexive, symmetric, asymmetric, antisymmetric, transitive, where R is a relation ob B & B={1,2,3,4}
(i) R={(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)}
(ii) R={(1,1),(2,2),(3,3)}
(iii) R={(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)}
(iv) R={}
(v) R=BXB
(vi)R={(1,2),(1,3),(3,1),(1,1),(3,3),(3,2),(1,4),(4,2),(3,4)}

Problem 28: If B={1,2,3,4,5} & R is a relation on B given by

(i)



(ii)



Find if the above relations are reflexive, symmetric, asymmetric, antisymmetric, transitive.

Problem 29: If R is a relation on B & B={a,b,c,d} & the matrix of R is given by below. Find which of the matrices are reflexive, symmetric, asymmetric, antisymmetric, transitive.

(i)

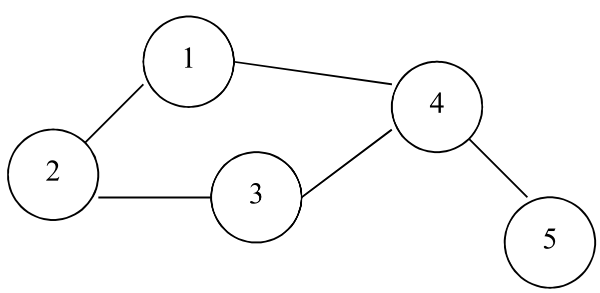
$$M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(ii)

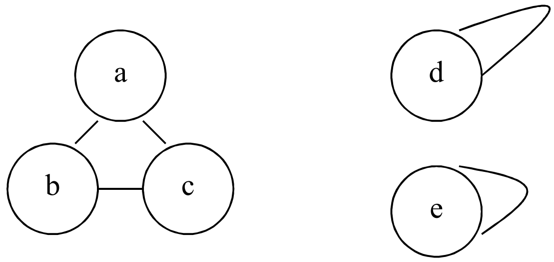
$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 30: Find R given:

(i)



(ii)



Problem 31: Draw the graph of R if R is a symmetric relation
R={(a,b),(b,a),(a,c),(c,a),(d,a),(a,d),(a,a),(b,b)}
where R is a relation on B & B={a,b,c,d}

More Examples:

Example 38: Determine if R is reflexive, symmetric, asymmetric, antisymmetric where A=Z & aRb if & only if $a \leq b+2$

Example 39: If A = Z & aRb if & only if a+b is even. Determine if R is reflexive, symmetric, asymmetric, antisymmetric, transitive.

Example 40: If S={1,2,3,4}
A = S X S
& R is a relation on A such that (a,b) R (c,d) if & only if ad = bc. Determine if R is reflexive, symmetric, asymmetric, antisymmetric, transitive.

Try it yourself

Problem 32: If $A=Z^+$, aRb if & only if $a=b^k$ for some $k \epsilon Z^+$. Determine if R is reflexive, symmetric, asymmetric, antisymmetric, transitive.

Problem 33: If A=Z & aRb if & only if $|a - b| = 2$ where R is a relation on A. Determine if R is reflexive, symmetric, asymmetric, antisymmetric, transitive.

Problem 34: If A is a set of all lines in a plane & aRb if & only if a is parallel to b where R is a relation on A. Determine if R is reflexive, symmetric, asymmetric, antisymmetric, transitive.

Problem 35: Let A = set of all ordered pairs of real numbers. (a,b)R(c,d) if & only if a = c, where R is a relation on A. Determine if R is reflexive, symmetric, asymmetric, antisymmetric, transitive.

More Examples:

Example 41: Show that if a relation on set A is transitive & irreflexive then it is asymmetric.

Try it yourself

Problem 36: Prove that if R is a relation on set A & is symmetric then R^2 is also symmetric.

Problem 37: Let R be a non – empty relation on set A. Suppose that R is symmetric & transitive, then show that R is not irrreflexive.

Equivalence Relation If a relation is reflexive, symmetric & transitive then it is **an equivalence relation**.

Try it yourself

Problem 38: Identify if the following is an equivalence relation or not

(i)

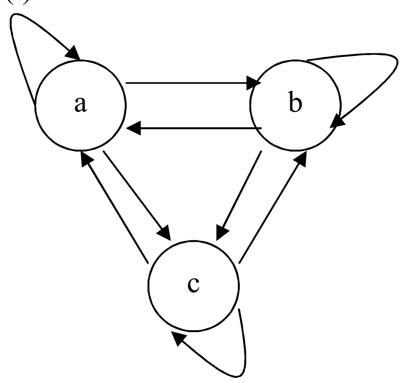
$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(ii)

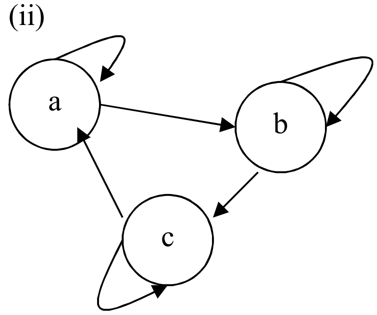
$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 39: Determine which of the following is a equivalence relation.

(i)



(ii)



Problem 40: Determine if R is an equivalence relation where R is a relation on A &

(i) A={1,2,3,4}
R={(1,1),(2,1),(2,2),(3,3),(4,4),(4,3)}
(ii) A={a,b,c,d,e}
R={(a,a),(a,b),(a,c),(b,a),(b,b),(c,a),(b,c),(c,c),(d,d),(c,b), (e,e)}
(iii) A={a,b,c,d}
R={(a,a),(a,b),(b,a),(b,b),(c,a),(c,c),(a,c),(d,a),(d,d)}

Problem 41: If A = set of all triangles & aRb if & only if a is similar to b, where R is a relation on A. Determine if R is an equivalence relation or not.

Problem 42: If A = $Z^+ \times Z^+$ and R is a relation on A such that (a,b)R(c,d) if & only if b = d. Determine if R is reflexive, symmetric, asymmetric, antisymmetric, transitive, equivalence relation.

Equivalence Relation from Partitions or equivalence classes

If P is the partition of a set A & R is the equivalence relation required on A then we can define R as aRb if & only if a & b are members of the same block (same set of the partition).

Theorem 3: If P is the partition of a set A & R is the equivalence relation required on A then we can define R as aRb if & only if a & b are members of the same block (same set of the partition).

Proof: aRa means a & a are members of the same block, which is obviously true.
∴ R <u>is reflexive.</u>
aRb means a & b are members of the same block.
∴ b & a are also members of the same block
∴ bRa is also true
∴ R <u>is symmetric.</u>
If aRb & bRc means a & b are members of the same block & b & c are also members of the same block, i.e. a,b & c all belong to the same block.
∴ a & c belong to the same block
∴ aRc is true.
∴ R <u>is transitive.</u>
∴ R is an **equivalence relation determined by the partition P**.

Example 42: Let A={a,b,c,d} and consider the partition P={{a,b,c},{d}} of A. Find the equivalence relation R on A determined by P.

Note: Since a, b & c all belong to the same block R(a)=R(b)=R(c)

Note: R(a) is sometimes written as [a] & is called **equivalence classes** of R.
∴ The partition consists of all equivalence classes of r & this partition is denoted by A/R. i.e. P is a quotient set of A that is constructed from & determines R.

Example 44: If A={1,2,3,4} & R is a equivalence relation given as
R={(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)}
Determine the equivalence classes & A/R.

∴ Steps involved in finding A/R are

- Choose an element 'a' of set A & find equivalence class R(a).
- If R(a) ≠ A, then choose another element 'b' from A such that $b \notin R(a)$ & then find R(b)
- If the union of the equivalence classes obtained so far is not equal to A then choose another element $c \notin$ union of the equivalence classes obtained so far & find R(c).
- Repeat the above step until all elements of A are included in the computed equivalence classes.

Note: If A is an infinite set then the process goes on indefinitely. But, we stop at a point when we emerge with a pattern.

Example 45: Let B={1,2,3,4,5} & A=BXB & R be a relation on A such that (a,b)R(a',b') if & only if ab'= a'b.
Show that R is an equivalence relation & find A/R.

Try it yourself

Problem 43: If P={{a,c,e},{b,d,f}} is a partition of A={a,b,c,d,e,f}. Find the equivalence relation on A.

Problem 44: Let B={1,2,3,4} & let A= BXB. Let R be a relation on A such that (a,b)R(a',b') if & only if a+b = a'+b'
(i) Show that R is an equivalence relation
(ii) Compute A/R.

Example 45: A relation R on set A is called circular if aRb & bRc implies cRa. Show that R is reflexive & circular if & only if it is an equivalence relation.

Ans: Let us assume R is reflexive & circular, then we shall show that it an equivalence relation.
Say aRb then bRb (since reflexive)
Since R is circular therefore the above statement will also imply bRa.
Thus, if aRb then bRa
∴ R is symmetric.
If aRb & bRc then cRa (since R is circular)
But R is also symmetric
∴ cRa implies aRc.
∴ aRb & bRc implies aRc
∴R is a transitive relation.
Since we have shown R to be reflexive, symmetric & transitive ∴ R is an equivalence relation.
∴ If R is reflexive & circular then it is an equivalence relation. ………….(I)

Now we assume R to be an equivalence relation & show that it is reflexive & circular.
Since R is an equivalence relation it is obviously reflexive.
Since R is an equivalence relation
∴ aRb & bRc then aRc
But R is also symmetric
∴ aRc implies cRa
∴ aRb & bRc then cRa
∴ R is circular.
∴If R is an equivalence relation the it is reflexive & circular. ………….(II)

From I & II we conclude that
R is reflexive & circular if & only if it is an equivalence relation

Manipulation Of Relations:

Complementary relation ($\bar{R}$): a $\bar{R}$ b if & only if a$\not R$b
a(R∩S)b if & only if aRb & bRa
a(RUS)b if & only if aRb or aSb
A(R⊕S) means aRb & a$\not S$b or a$\not R$b & aSb
Inverse relation (R^{-1}): bR^{-1}a if & only if aRb
i.e. (a,b)ϵR then (b,a)ϵR^{-1}.
$(R^{-1})^{-1} = R$
$Dom(R^{-1}) = Ran(R)$
$Ran(R^{-1}) = Dom(R)$

Example 46: Let A={1,2,3,4} & B={a,b,c} & R & S be relations from A to B such that
R={(1,a),(1,c),(2,a),(2,c),(3,b),(4,b),(4,c)}
S={(1,b),(2,c),(3,a),(3,b),(4,b),(4,c)}
Find (a) $\bar{R}$ (b) R∩S (c) RUS (d) R^{-1}

Try it yourself
Problem 45:

Find $\overline{R}$, $\overline{S}$, $R \cap S$, $R \cup S$, R^{-1}, S^{-1} if
A=B={1,2,3} & R & S are relations from A to B given by
R={(1,1),(1,2),(2,3),(3,1)}
S={(2,1),(3,1),(3,2),(3,3)}

Problem 46:

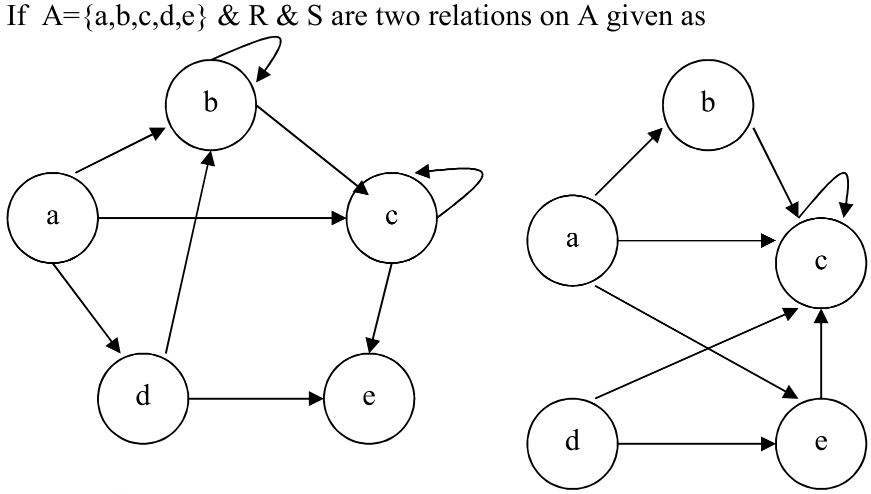
Find $\overline{R}$, $\overline{S}$, $R \cap S$, $R \cup S$, R^{-1}, S^{-1} if
A={1,2,3} & B={a,b,c} & R & S are relations from A to B given by
R={(1,a),(2,a),(3,b),(3,c)}
& S={(1,a),(1,b),(2,a),(2,b)}

In a **digraph**

- $\overline{R}$ is all the possible edges that are missing.
- $R \cap S$ are all the edges common in the diagraph of R & S.
- $R \cup S$ are all the edges of R & S.
- $R \oplus S$ means only those edges of R which are not in S OR those edges of S that are not in R.
- R^{-1} is obtained by inverting all the edges directions.

Example 47:

If A={a,b,c,d,e} & R & S are two relations on A given as

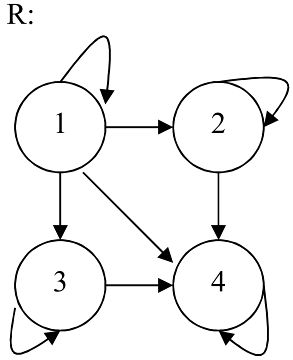


Find $\overline{R}$, $\overline{S}$,R^{-1}, S^{-1}, $R \cup S$, $R \cap S$.

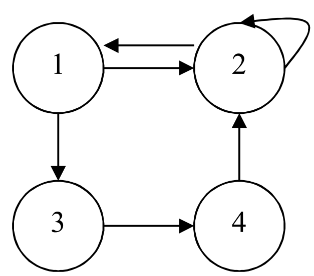
Try it yourself
Problem 47:

Find $\overline{R}$, $\overline{S}$,R^{-1}, S^{-1}, $R \cup S$, $R \cap S$ of

R:



S:



For a **matrix** M_R , $M_{\overline{R}}$ is obtained by replacing all 0's by 1's & all 1's by 0's.
$M_{R^{-1}}$ is nothing but M_R^T
$M_{R \cup S} = M_R \vee M_S$
$M_{R \cap S} = M_R \wedge M_S$

Try it yourself

Problem 48: Let A = {a,b,c}

B = {1,2,3,4}

Let R & S be two relations from A to B such that

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Find $\bar{R}$, $\bar{S}$, R^{-1}, S^{-1}, RUS, R∩S

Problem 49: If

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find $M_{\bar{R}}$, $M_{\bar{S}}$, M_R^{-1}, M_S^{-1}

Problem 50: Let A = B = {1,2,3,4}

R={(1,1),(1,2),(1,3),(3,1),(4,2),(4,3),(4,4)}

S={(1,1),(2,3),(3,4),(4,3),(4,4)}

Find $M_{\bar{R}}$, $M_{\bar{S}}$, M_R^{-1}, M_S^{-1}, M_{RUS}, $M_{R\cap S}$

Example 48: If A={1,2,3,4,5,6}

& R & S are two equivalence relations on A where

R={(1,2),(1,1),(2,1),(2,2),(3,3),(4,4),(5,5),(5,6),(6,5),(6,6)}

S={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,6),(4,4),(6,4),(6,6),(5,5)}

Compute the partition of R∩S.

Try it yourself

Problem 51: Let A={1,2,3,4,5} & let R & S be two equivalence relations on A where M_R & M_S are as given below:

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Compute the partition of A corresponding to R∩S

Closures:

If R is a relation on A & say it does not possess a given property (e.g. reflexivity). Say we want to make R posses that property by adding as few pairs as possible. The smallest relation thus obtained (say R_1 on A), which posses that property desired is called **closure** of R with respect to the property in question.

e.g. Say R is a relation on A which is not reflexive then
$R_1 = R \cup \Delta$
where $\Delta = \{(a,a) | a \in A\}$
is called the reflexive closure of R.

e.g. If R is not symmetric i.e. there exists some $(x,y) \in R$ such that $(y,x) \notin R$.
If $(x,y) \in R$ then $(y,x) \in R^{-1}$
$\therefore$ $R \cup R^{-1}$ will be symmetric $[(R \cup R^{-1})^{-1} = R \cup R^{-1}]$
$\therefore R \cup R^{-1}$ is the symmetric closure of R.

Example 49:
If A={1,2,3,4} &
R={(1,2),(2,3),(1,3),(3,4)}
Find the reflexive & symmetric closure of R.

In terms of a **diagraph** a reflexive closure will have cycles of length 1 from each vertex & a symmetric closure will have all edges bidirectional (i.e. a graph).

Composition of Relations

Let R be a relation from A to B & S be a relation from B to C, then $S \circ R$ is a relation from A to C such that $a(S \circ R)b$ if & only if there is some b for which aRb & bSc.

Example 50:
Let A = {a,b,c,d}
R = {(a,a),(a,b),(a,c),(b,d),(c,b)}
S = {(a,d),(a,c),(b,c),(c,a),(d,a)}
Find $S \circ R$.

Note: The relative set
$(S \circ R)(A_1) = S(R(A_1))$

Try it yourself:

Problem 52:
Let A={1,2,3,4}
R = {(1,1),(1,2),(2,3),(2,4),(3,4),(4,1),(4,3)}
S = {(3,1),(4,4),(2,3),(2,4),(1,1),(1,4)}
Find $S \circ R$, $R \circ R$, $S \circ S$, $R \circ S$.

Problem 53:
If R & S are equivalence relations then is $S \circ R$ an equivalence relation.

In terms of matrix of a relation $M_{S \circ R}$ is obtained by identifying a 1 in say row 'a' column 'b' in R then a 1 in row 'b' column 'c' of S, resulting in a 1 in row 'a' column 'c' of $M_{S \circ R}$.
A closer look at the logic of finding $M_{S \circ R}$ reveals an interesting fact that
$M_{S \circ R} = M_R \odot M_S$

Example 51: If A = {1,2,3} &

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find $M_{S\circ R}$ hence find $S\circ R$

Example 52: Let R be a relation from A to B & let S and T be relations from B to C. Prove or disapprove $(S \cup T)\circ R = (S\circ R)\cup(T\circ R)$

Try it yourself:

Problem 54: Compute $M_{S\circ R}$, $M_{R\circ S}$, $M_{S\circ S}$, $M_{R\circ R}$ & hence find $S\circ R$, $R\circ S$, $S\circ S$, $R\circ R$

(a)

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Problem 55: Let R be a relation from A to B & let S and T be relations from B to C. Prove or disapprove $(S \cap T)\circ R = (S\circ R)\cap(T\circ R)$

Problem 56: Let R be a relation from A to B & let S and T be relations from B to C. Show that if $R \subseteq S$, then $T\circ R \subseteq T\circ S$

Some important characteristics

- $T\circ(S\circ R) = (T\circ S)\circ R$
- $M_{T\circ(S\circ R)} = M_{S\circ R} \odot M_T = (M_R \odot M_S) \odot M_T$
- $M_{(T\circ S)\circ R} = M_R \odot (M_S \odot M_T)$
- $(M_R \odot M_S) \odot M_T = M_R \odot (M_S \odot M_T)$
- $M_{T\circ(S\circ R)} = M_{(T\circ S)\circ R}$
- $T\circ(S\circ R) = (T\circ S)\circ R$
- $S\circ R \neq R\circ S$
- $(S\circ R)^{-1} = R^{-1}\circ S^{-1}$

Transitive Closure & Warshall's Algorithm:

Transitive Closure

R^∞ is the transitive closure of R i.e. R^∞ is the smallest transitive relation containing R. (Theorem in e*theo supplement)

Finding R^∞

Method I

Via digraph ….. done earlier

The disadvantage of the above technique is that it is not methodological & can get very impractical for large sets & relations.

Method II

If A is a set such that |A| = n then

$M_{R\infty} = M_R V(M_R \odot M_R) V(M_R \odot M_R \odot M_R)\ldots..(M_R \odot MR \odot M_R \odot \ldots$ n terms)

e.g. if n = 3 then

$M_{R\infty} = M_R V(M_R \odot M_R) V(M_R \odot M_R \odot M_R)$

(Theorem in e*theo supplement)

The above technique though methodological is inefficient for large matrices.

Example 53:

If A={1,2,3,4}

R = {(1,2),(2,3),(3,4),(2,1)}

Find the transitive closure of R.

Warshall's Algorithm to find transitive closure

Let $\mathbf{W_0} = M_R$

Find $\mathbf{W_1}$ as follows

- Copy all 1's of W_0 into W_1
- Look in W_0 at column 1 & note down the positions at which a 1 occurs. Then look at row 1 & note down the positions at which a 1 occurs. Not put a 1 in W_1 at all possible rows, columns obtained from the foregoing analysis. e.g. on monitoring column 1, 1's were at position 3 & 5; on monitoring row 1 say 1's were at position 2, 3 & 4, then we put a 1 in W_1 at row 3 column 2, 3 & 4 & also in row 5 column 2, 3 & 4.

Find $\mathbf{W_2}$ (same method as above) but now we look at column 2 & row 2 of W_1.

and so on ……………. Till W_n where n is the number of elements in A & R is a relation on A.

(For theoretical proof of Warshall's Algorithm see e*theo)

Example 54:

Redo example 53 using Warshall's algoritm.

Example 55:

$$M_R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Where R & S are relations on A={1,2,3,4,5}

Find the smallest equivalence relation containing R & also find the partition of A that it produces.

Try it yourself:

Problem 57:

Let A={a,b,c,d}
Find the transitive closure by Warshall's algorithm

(a)

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

(c)

$$M_R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Problem 58:

If A = {1,2,3,4,5}
& R & S are equivalence relations on A. Find the smallest equivalence relation containing R & S & list the elements of this relation & the partition of A corresponding to the equivalence relations found.

(a)

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note: If R & S are two equivalence relations on A then RUS will also be reflexive & symmetric but may not be transitive.

Problem 59: Let A = {a,b,c,d,e} & let R & S be relations on A given by

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$M_S = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Using Warshall's algorithm compute the transitive closure of RUS.