Q1) (i) $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

$= \int_0^1 dx \int_0^{\sqrt{1+x^2}} \frac{1}{y^2+(1+x^2)} \cdot dy$

$= \int_0^1 dx \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}}$

$= \int_0^1 dx \left[\frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4}\right] = \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot dx$

$= \frac{\pi}{4} \left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1 = \frac{\pi}{4} \log\left(1+\sqrt{2}\right)$

Q2) (ii) $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\,dy\,dx$

$= \int_0^a x^2 dx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$

$= \int_0^a x^2 \left(\frac{a^2-x^2}{2}\right) dx = \frac{1}{2} \int_0^a (a^2x^2-x^4)dx$

$$= \frac{1}{2}\left[\frac{a^2x^3}{3} - \frac{x^5}{5}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^5}{3} - \frac{a^5}{5}\right] = \frac{1}{2}\cdot\frac{2a^5}{15} = \frac{a^5}{15}$$

Q3) ~~(ii)~~ $I = \int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$

$$= \int_0^1 dx \int_0^{x^2} e^{y/x}\cdot dy$$

$$= \int_0^1 dx \left[x e^{y/x}\right]_0^{x^2} = \int_0^1 (x e^x - x)\, dx$$

$$= \left[x e^x - e^x - \frac{x^2}{2}\right]_0^1 = \left[e - e - \frac{1}{2} + 1\right]$$

$$= 1/2$$

Q4) (iv) $I = \int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \left\{\frac{x\, dy}{y^2 + (x^2+a^2)}\right\} dx$

$$= \int_0^{a\sqrt{3}} dx \cdot x \int_0^{\sqrt{x^2+a^2}} \frac{dy}{y^2+(x^2+a^2)}$$

$$= \int_0^{a\sqrt{3}} x\,dx \left[\frac{1}{\sqrt{x^2+a^2}} \tan^{-1}\left(\frac{y}{\sqrt{x^2+a^2}}\right)\right]_0^{\sqrt{x^2+a^2}}$$

$$= \int_0^{a\sqrt{3}} x\,dx \left[\frac{1}{\sqrt{x^2+a^2}} \cdot \frac{\pi}{4}\right]$$

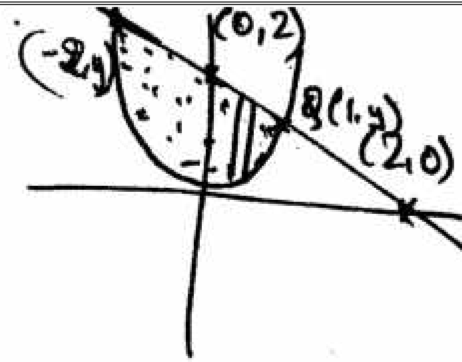
$$= \frac{\pi}{4\cancel{2}} \int_0^{a\sqrt{3}} \frac{\cancel{2}x}{\sqrt{x^2+a^2}}\,dx$$

Let $x^2+a^2=t^2$

$\therefore \cancel{2}x\,dx = \cancel{2}t\,dt$

x	0	$a\sqrt{3}$
t	a	$2a$

$$I = \frac{\pi}{4}\int_a^{2a} \frac{\cancel{t}\,dt}{\cancel{\sqrt{t^2}}} = \frac{\pi}{4}\left[t\right]_a^{2a} = \frac{\pi a}{4}$$



5) 1 i) $\iint y\,dx\,dy$

For Q: $x^2+x-2=0$

$$\therefore x = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm 3}{2} = 1 \text{ or } -2$$

~~I =~~ Limits: $y=x^2$; $y=2-x$

$x=-2$; $x=1$

$$I = \int_{-2}^{1} dx \int_{x^2}^{2-x} y\,dy = \int_{-2}^{1} dx \left[\frac{y^2}{2}\right]_{x^2}^{2-x}$$

$$I = \int_{-2}^{1} dx \left[\frac{(2-x)^2}{2} - \frac{x^4}{2}\right]$$

$$= \cancel{\int_{-2}^{1} \left(\frac{4-4x+x^2-x^4}{2}\right) dx}$$

$$= \cancel{\int_{-2}^{1} (2-2x)\,dx = 2\int_0^1 (1-x)\,dx}$$

$$= \cancel{2\left[x - \frac{x^2}{2}\right]_{-2}^{1} = 2\left(1-\frac{1}{2}\right) = 2\cdot\frac{1}{2}}$$

$$= \cancel{2\left(1-\frac{1}{2}+2+\frac{4}{2}\right) = 2\cdot\left(\frac{9}{2}\right)}$$

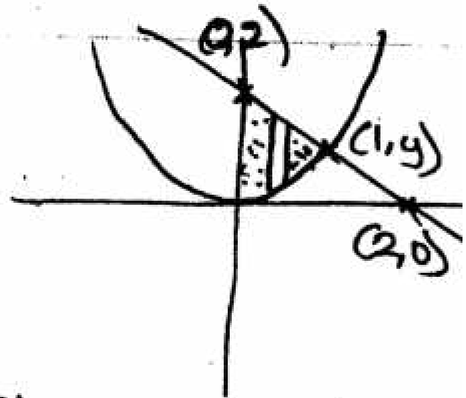
$$= \frac{1}{2}\int_{-2}^{1} \left[(2-x)^2 - x^4\right] dx$$

$$= \frac{1}{2}\left[\frac{(2-x)^3}{-3} - \frac{x^5}{5}\right]_{-2}^{1}$$

$$= \frac{1}{2}\left[-\frac{1}{3} - \frac{1}{5} + \frac{64}{3} - \frac{32}{5}\right]$$

$$= \frac{1}{2}\left[\frac{63}{3}^{21} - \frac{33}{5}\right] = \frac{1}{2}\left(\frac{72}{5}\right) = \frac{36}{5}$$

6) ii) $I = \int_0^1 dx \int_{x^2}^{2-x} y\,dy$



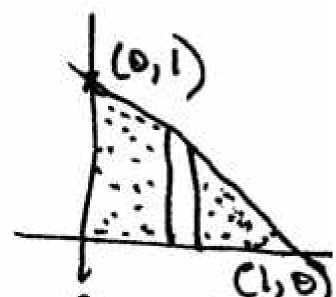
$= \int_0^1 dx \left[\frac{y^2}{2}\right]_{x^2}^{(2-x)}$

$= \frac{1}{2}\int_0^1 \left[(2-x)^2 - x^4\right] dx$

$= \frac{1}{2}\left[\frac{(2-x)^3}{-3} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2}\left[-\frac{1}{3} - \frac{1}{5} + \frac{8}{3}\right]$

$= \frac{1}{2}\left(\frac{7}{3} - \frac{1}{5}\right) = \frac{1}{2}\left(\frac{32}{15}\right) = \frac{16}{15}$

7 $I = \iint xy\sqrt{1-x-y}\,dx\,dy$



Limits: $y=0$; $y=1-x$

$x=0$; $x=1$

$I = \int_0^1 x\,dx \int_0^{1-x} y\sqrt{1-x-y}\,dy$ $\left\{\because \int_0^a f(t)\,dt = \int_0^a f(a-t)\,dt\right\}$

$= \int_0^1 x\,dx \int_0^{1-x} (1-x-y)\sqrt{\cancel{1}-\cancel{x}-\cancel{1}+\cancel{x}+y}\,dy$

$$I = \int_0^1 x\,dx \int_0^{1-x} \left(y^{1/2} - x y^{1/2} - y^{3/2}\right) dy$$

$$= \int_0^1 x\,dx \left[\frac{y^{3/2}}{3/2} - \frac{x y^{3/2}}{3/2} - \frac{y^{5/2}}{5/2}\right]_0^{1-x}$$

$$= 2\int_0^1 x\,dx \left[\frac{(1-x)^{3/2}}{3} - \frac{x(1-x)^{3/2}}{3} - \frac{(1-x)^{5/2}}{5}\right]$$

$$= \frac{2}{3}\int_0^1 x(1-x)^{3/2}\,dx - \frac{2}{3}\int_0^1 x^2(1-x)^{3/2} - \frac{2}{5}\int_0^1 x(1-x)^{5/2}$$

$$= \frac{2}{3}\beta\left(2,\frac{5}{2}\right) - \frac{2}{3}\beta\left(3,\frac{5}{2}\right) - \frac{2}{5}\beta\left(2,\frac{7}{2}\right)$$

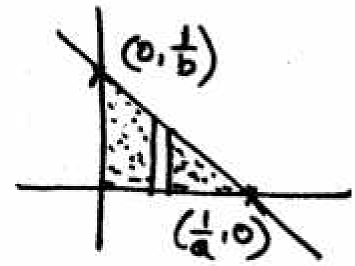
$$= \frac{2}{3}\frac{\Gamma 2\,\Gamma\frac{5}{2}}{\Gamma\frac{9}{2}} - \frac{2}{3}\frac{\Gamma 3\,\Gamma\frac{5}{2}}{\Gamma\frac{11}{2}} - \frac{2}{5}\frac{\Gamma 2\,\Gamma\frac{7}{2}}{\Gamma\frac{11}{2}}$$

$$= \frac{2}{3}\frac{1.\cancel{\Gamma\frac{5}{2}}}{\frac{7}{2}\cdot\frac{5}{2}\cancel{\Gamma\frac{5}{2}}} - \frac{2}{3}\frac{2\cancel{\Gamma\frac{5}{2}}}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cancel{\Gamma\frac{5}{2}}} - \frac{2}{5}\frac{1\cancel{\Gamma\frac{7}{2}}}{\frac{9}{2}\cdot\frac{7}{2}\cancel{\Gamma\frac{7}{2}}}$$

$$= \frac{8}{105} - \frac{32}{945} - \frac{8}{315}$$

$$= \frac{72-32-24}{945} = \frac{72-56}{945} = \frac{16}{945}$$

8) $I = \iint e^{ax} \cdot e^{by}\, dx\, dy$



Limits: $y = 0$; $y = \frac{1-ax}{b}$

$x = 0$; $x = 1/a$

$$I = \int_0^{1/a} e^{ax}\, dx \int_0^{\frac{1-ax}{b}} e^{by}\, dy$$

$$= \int_0^{1/a} e^{ax}\, dx \left[\frac{e^{by}}{b}\right]_0^{\frac{1-ax}{b}}$$

$$= \int_0^{1/a} e^{ax}\, dx \left[\frac{e^{(1-ax)}}{b} - \frac{1}{b}\right]$$

$$= \frac{1}{b}\int_0^{1/a} \left[e^{1} \cdot e^{-ax} - 1\right] e^{ax}\, dx$$

$$= \frac{1}{b}\int_0^{1/a} (e - e^{ax})\, dx = \frac{1}{b}\left[xe - \frac{e^{ax}}{a}\right]_0^{1/a}$$

$$I = \frac{1}{b}\left[\frac{e}{a} - \frac{e}{a} + \frac{1}{a}\right] = \frac{1}{ab}$$

9 ii)
$$I = \int_0^{1/a} dx \int_0^{\frac{1-ax}{b}} \sin\pi(ax+by)\,dy$$

$$= \int_0^{1/a} dx \int_0^{\frac{1-ax}{b}} \sin(\pi ax+\pi by)\,dy$$

$$= \int_0^{1/a} dx \left[-\frac{\cos(\pi ax+\pi by)}{\pi b}\right]_0^{\frac{1-ax}{b}}$$

$$= -\frac{1}{\pi b}\int_0^{1/a}\left\{\cos\left[\pi ax + \pi \cancel{b}\left(\frac{1-ax}{\cancel{b}}\right)\right] - \cos\pi ax\right\}dx$$

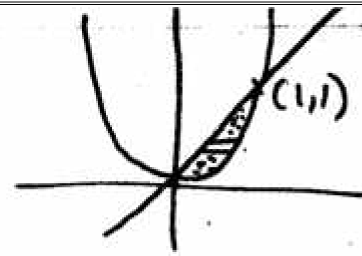
$$= -\frac{1}{\pi b}\int_0^{1/a}\left\{\cos\left[\cancel{\pi ax}+\pi-\cancel{\pi ax}\right]-\cos\pi ax\right\}dx$$

$$= -\frac{1}{\pi b}\int_0^{1/a}(-1-\cos\pi ax)\,dx$$

$$= \frac{1}{\pi b}\left[x+\frac{\sin\pi ax}{\pi a}\right]_0^{1/a} = \frac{1}{\pi b}\left[\frac{1}{a}+\frac{\sin\pi}{\pi a}\right]$$

$$= \frac{1}{\pi ab}$$

10-b. $I = \iint xy(x+y)\,dx\,dy$



Limits: $x = y$; $x = \sqrt{y}$

$y = 0$; $y = 1$

$$I = \int_0^1 y\,dy \int_y^{\sqrt{y}} x(x+y)\,dx$$

$$= \int_0^1 y\,dy \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_y^{\sqrt{y}}$$

$$= \int_0^1 y\,dy \left[\frac{y^{3/2}}{3} + \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^3}{2}\right]$$

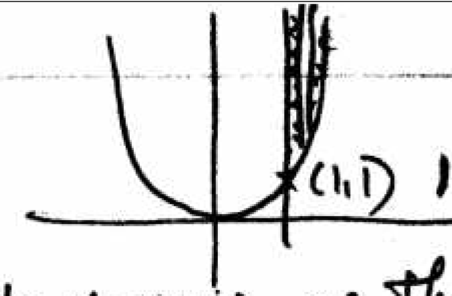
$$= \int_0^1 \left[\frac{y^{5/2}}{3} + \frac{y^3}{2} - \frac{y^4}{3} - \frac{y^4}{2}\right] dy$$

$$= \left[\frac{2y^{7/2}}{21} + \frac{y^4}{8} - \frac{y^5}{15} - \frac{y^5}{10}\right]_0^1$$

$$= \frac{2}{21} + \frac{1}{8} - \frac{1}{15} - \frac{1}{10} = \frac{2}{21} + \frac{1}{8} - \frac{\cancel{5}\,1}{\cancel{30}\,6}$$

$$= \frac{37}{168} - \frac{1}{6} = \frac{222-168}{168\times 6} = \frac{222-168}{168\times 6} = \frac{\cancel{54}\,3}{\cancel{168\times 6}\,56} = \frac{3}{56}$$

11 7 $I = \iint \dfrac{dx\,dy}{x^4+y^2}$



Strip is taken // to y-axis as the integrand can be int. wrt y

Limits :

$$y = x^2 \quad ; \quad y = \infty$$
$$x = 1 \quad ; \quad x = \infty$$

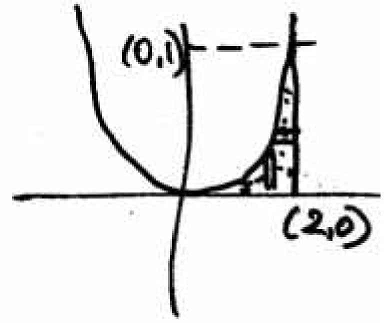
$$I = \int_1^\infty dx \int_{x^2}^{\infty} \frac{1}{y^2+x^4}\, dy$$

$$= \int_1^\infty dx \left[\frac{1}{x^2} \tan^{-1} \frac{y}{x^2} \right]_{x^2}^{\infty}$$

$$= \int_1^\infty dx \left[\frac{1}{x^2} \cdot \frac{\pi}{2} - \frac{1}{x^2} \cdot \frac{\pi}{4} \right]$$

$$= \int_1^\infty dx \left[\frac{1}{x^2} \cdot \frac{\pi}{4} \right] = \frac{\pi}{4} \int_1^\infty \frac{dx}{x^2}$$

$$= \frac{\pi}{4} \left[\frac{-1}{x} \right]_1^\infty = \frac{\pi}{4} \left[0 + 1 \right] = \frac{\pi}{4}$$



12 $I = \int_0^2 \int_0^{x^2/4} xy \, dx \, dy$

Limits: $y = 0$; $4y = x^2$
$x = 0$; $x = 2$

Changing order of int.

Limits : $x = 2\sqrt{y}$; $x = 2$

$y = 0$; $y = 1$

$$I = \int_0^1 y\,dy \int_{2\sqrt{y}}^{2} x\,dx$$

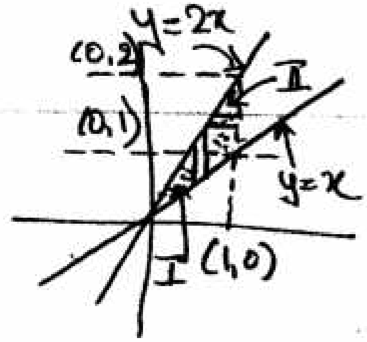
$$= \int_0^1 y\,dy \left[\frac{x^2}{2}\right]_{2\sqrt{y}}^{2}$$

$$= \int_0^1 y\,dy\,(2 - 2y)$$

$$= 2\int_0^1 (y - y^2)\,dy = 2\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

19. $\int_0^1 \int_x^{2x} dx\,dy$



Limits: $y = x$; $y = 2x$

$x = 0$; $x = 1$

Changing order of int.

Region I: Limits: $x = y/2$; $x = y$

$y = 0$; $y = 1$

$$I_1 = \int_0^1 dy \int_{y/2}^{y} dx = \int_0^1 dy \left[x\right]_{y/2}^{y}$$

$$= \int_0^1 \frac{y}{2}\, dy = \left[\frac{y^2}{4}\right]_0^1 = \frac{1}{4}$$

Region II: Limits: $x = y/2$; $x = 1$

$y = 1$; $y = 2$

$$I_2 = \int_1^2 dy \int_{y/2}^{1} dx = \int^2 dy \left[x\right]_{y/2}^{1}$$

$$= \int_1^2 \left(-\frac{y}{2} + 1\right) dy = \left[-\frac{y^2}{4} + y\right]_1^2$$

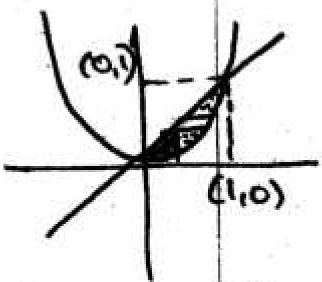
$$= \left(\cancel{-1} + \cancel{2} + \frac{1}{4} - \cancel{1}\right) = +\frac{1}{4}$$

$$\therefore I = I_1 + I_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

11. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} xy\,dx\,dy$

Limits: $x=y$; $x=\sqrt{y}$ $\therefore$ $y=x^2$

$y=0$; $y=1$



Changing order of int.

Limits: $y=x^2$; $y=x$

$x=0$; $x=1$

$$I = \int_0^1 x\,dx \int_{x^2}^{x} y\,dy$$

$$= \int_0^1 x\,dx \left[\frac{y^2}{2}\right]_{x^2}^{x} = \int_0^1 x\,dx\left(\frac{x^2}{2}-\frac{x^4}{2}\right)$$

$$= \frac{1}{2}\left[\frac{x^4}{4}-\frac{x^6}{6}\right]_0^1 = \frac{1}{2}\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{24}$$

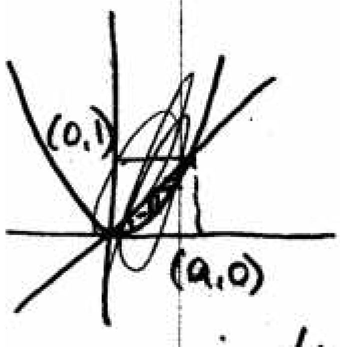
$$= \frac{1}{24}$$

2hrs B-I

12 $\displaystyle\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy = \int_0^a \int_{\sqrt{x/a}}^{x/a} \cancel{(x^2+y^2)\,dx\,dy}$

Limits: $y=\frac{x}{a}$ $\therefore x=ay$;

$y=\sqrt{\frac{x}{a}}$ $\therefore x=ay^2$



$x=0$; $x=a$

Changing order of int.

$\therefore$ Limits: $x=ay$; $x=ay^2$

$y=0$; $y=1$

$$I=+\int_0^1 dy\int_{ay^2}^{ay}(x^2+y^2)\,dx$$

$$=+\int_0^1 dy\left[\frac{x^3}{3}+xy^2\right]_{ay^2}^{ay}$$

$$=+\int_0^1\left(-\frac{a^3y^6}{3}-ay^4+\frac{a^3y^3}{3}+ay^3\right)dy$$

$$=+\left[-\frac{a^3y^7}{21}-\frac{ay^5}{5}+\frac{a^3y^4}{12}+\frac{ay^4}{4}\right]_0^1$$

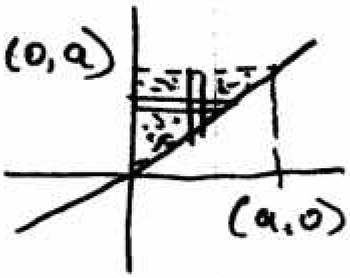
$$=+\left(-\frac{a^3}{21}-\frac{a}{5}+\frac{a^3}{12}+\frac{a}{4}\right)=+\left(+\frac{a}{20}+\frac{3a^3}{7\times3\times4}\right)=\frac{a}{20}+\frac{a^3}{28}$$

13. $$I = \int_0^a \int_0^y \frac{x\,dx\,dy}{\sqrt{(a^2-x^2)(a-y)(y-x)}}$$

Limits:

$x = 0 \;;\; x = y$

$y = 0 \;;\; y = a$



Changing order of int.

Limits:

$y = x \;;\; y = a$

$x = 0 \;;\; x = a$

$$I = \int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}} \int_x^a \frac{dy}{\sqrt{(y-x)(a-y)}}$$

Let $I_1 = \int_x^a \frac{dy}{\sqrt{(y-x)(a-y)}}$

y	x	a
t	0	1

Let $y - x = (a-x)t$

$\therefore a - y = (a-x)(1-t)$

$\therefore dy = (a-x)dt$

$$I_1 = \int_0^1 \frac{1}{\sqrt{(a-x)t\,.\,(a-x)(1-t)}}\,.(a-x)\,dt$$

$$= \int_0^1 t^{-1/2}(1-t)^{-1/2}\,dt$$

$$= \beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1} = \pi$$

$$\therefore I = \int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}}\,.\,\pi$$

Let $a^2 - x^2 = t^2$

$$\therefore -\cancel{2}x\,dx = \cancel{2}t\,dt$$

x	0	a
t	a	0

$$I = \pi\int_a^0 \frac{-t\,dt}{\sqrt{t^2}} = \pi\int_0^a dt = \pi[t]_0^a$$

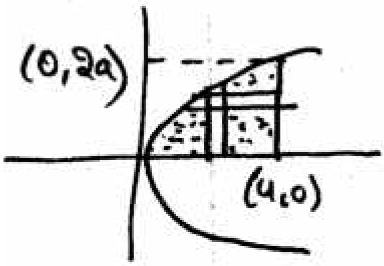
$$= \pi a$$

2-B-II

$$I = \int_0^a \int_0^{2\sqrt{xa}} x^2\,dx\,dy.$$

Limits: $y=0$; $y=2\sqrt{xa}$ $\therefore y^2=4ax$

$x=0$; $x=a$



Changing order of int.

$\therefore x=y^2/4a$; $x=a$

$y=0$; $y=2a$

$$I=\int_0^{2a}dy\int_{y^2/4a}^{a}x^2\,dx=\int_0^{2a}dy\left[\frac{x^3}{3}\right]_{y^2/4a}^{a}$$

$$=\int_0^{2a}\left(\frac{a^3}{3}-\frac{y^6}{64\cdot 3a^3}\right)dy=\frac{1}{3}\left[a^3y-\frac{y^7}{448a^3}\right]_0^{2a}$$

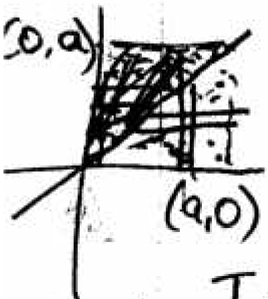
$$=\frac{1}{3}\left(2a^4-\frac{\cancel{64}\times 2a^{\cancel{7}4}}{\cancel{64}\times 7\cancel{a^3}}\right)=\frac{2}{3}a^4\left(1-\frac{1}{7}\right)$$

$$=\frac{2}{\cancel{3}}a^4\cdot\frac{\cancel{6}^{\,2}}{7}=\frac{4a^4}{7}$$

18. $I = \int_0^a \int_y^a \frac{x^2\,dy\,dx}{\sqrt{x^2+y^2}}$

Limits: $x = y$; $x = a$

$y = 0$; $y = a$



Changing order of int.

Limits: $y = 0$; $y = x$

$x = 0$; $x = a$

$$I = \int_0^a x^2\,dx \int_0^x \frac{dy}{\sqrt{y^2+x^2}}$$

$$= \int_0^a x^2\,dx \left[\log\left(y+\sqrt{y^2+x^2}\right)\right]_0^x$$

$$= \int_0^a x^2\,dx \left[\log\left(x+\sqrt{x^2+x^2}\right) - \log(x)\right]$$

$$\cancel{= \int_0^a x \log\left(\frac{a+\sqrt{a^2+x^2}}{x+\sqrt{2}\,x}\right)dx}$$

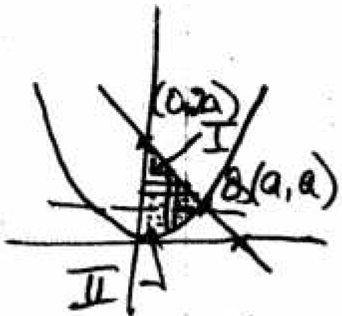
$$= \int_0^a x^2\,dx \log\left[\frac{\cancel{x}(1+\sqrt{2})}{\cancel{x}}\right]$$

$$= \log(1+\sqrt{2})\left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}\log(1+\sqrt{2})$$

14 $I = \int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$

Limits: $x^2 = ay$; $y = 2a - x$ ∴ $x + y = 2a$

$x = 0$; $x = a$



For pt. Q: $x^2 = 2a^2 - ax$

∴ $x^2 + ax - 2a^2 = 0$

$x = \frac{-a \pm \sqrt{a^2 + 8a^2}}{2} = \frac{-a \pm 3a}{2} = a$

Changing order of int.

Region II: Limits: $x = 0$; $x = \sqrt{ay}$

$y = 0$; $y = a$

$I = \int_0^a y\,dy \int_0^{\sqrt{ay}} x\,dx = \int_0^a y\,dy \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}}$

$= \int_0^a \frac{ay^2}{2}\,dy = \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a = \frac{a}{6}(a^3)$

Region I: Limits: $x = 0$; $x = 2a - y$

$y = a$; $y = 2a$

$I = \int_a^{2a} y\,dy \int_0^{2a-y} x\,dx = \int_a^{2a} y\,dy \left[\frac{x^2}{2}\right]_0^{2a-y}$

$$I = \frac{1}{2}\int_a^{2a} y(2a-y)^2\,dy$$

$$= \frac{1}{2}\int_a^{2a} y(4a^2 - 4ay + y^2)\,dy$$

$$= \frac{1}{2}\int_a^{2a} (4a^2y - 4ay^2 + y^3)\,dy$$

$$= \frac{1}{2}\left[\frac{4a^2y^2}{2} - \frac{4ay^3}{3} + \frac{y^4}{4}\right]_a^{2a}$$

$$= \frac{1}{2}\left(8a^4 - \frac{32a^4}{3} + \frac{16a^4}{4} - 2a^4 + \frac{4}{3}a^4 - \frac{1}{4}a^4\right)$$

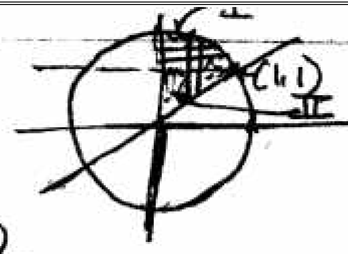
$$= \frac{a^4}{2}\left(10 - \frac{28}{3} - \frac{1}{4}\right)$$

$$= \frac{a^4}{2}\left[\frac{39}{4} - \frac{28}{3}\right] = \frac{a^4}{2}\left(\frac{117-112}{12}\right)$$

$$= \frac{a^4}{2}\left(\frac{5}{12}\right) = \frac{5a^4}{24}$$

$$I = I_1 + I_2 = \frac{5a^4}{24} + \frac{a^4}{6} = \frac{9a^4}{24} = \frac{3a^4}{8}$$

27. $I = \int_0^1 dx \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy$



Limits : $y = x$; $x^2+y^2 = 2$

$x = 0$; $x = 1$

Changing order of int.

Region I: Limits : $x = 0$; $x = \sqrt{2-y^2}$

$y = 1$; $y = \sqrt{2}$

$I_1 = \int_1^{\sqrt{2}} dy \int_0^{\sqrt{2-y^2}} \frac{x \, dx}{\sqrt{x^2+y^2}}$

Let $x^2+y^2 = t^2$

$\therefore x dx = t dt$

x	0	$\sqrt{2-y^2}$
t	y	$\sqrt{2}$

$I_1 = \int_1^{\sqrt{2}} dy \int_y^{\sqrt{2}} \frac{\cancel{t} dt}{\sqrt{\cancel{t^2}}} = \int_1^{\sqrt{2}} [\sqrt{2} - y] dy$

$= \left[\sqrt{2}\, y - \frac{y^2}{2}\right]_1^{\sqrt{2}} = \left(2 - \frac{2}{2} - \sqrt{2} + \frac{1}{2}\right)$

$= \left(\frac{3}{2} - \sqrt{2}\right) = \cancel{\left(\frac{3}{2} - \frac{2}{\sqrt{2}}\right)}$

Region II: Limits : $x = 0$; $x = y$
$y = 0$; $y = 1$

$$I_2 = \int_0^1 dy \cdot \int_0^y \frac{x\,dx}{\sqrt{x^2+y^2}}$$

Let $x^2+y^2 = t^2$

$\therefore x\,dx = t\,dt$

x	0	y
t	y	$y\sqrt{2}$

$$I_2 = \int_0^1 dy \int_y^{y\sqrt{2}} \frac{\cancel{t}\,dt}{\sqrt{\cancel{t^2}}} = \int_0^1 (y\sqrt{2}-y)\,dy$$

$$= (\sqrt{2}-1)\int_0^1 y\,dy = (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2}(\sqrt{2}-1) = \cancel{\frac{1}{\sqrt{2}} - \frac{1}{2}}$$

$$I = I_1 + I_2 = \frac{3}{2} - \sqrt{2} + \frac{\sqrt{2}}{2} - \frac{1}{2}$$

$$= 1 - \frac{\sqrt{2}}{2} = 1 - \frac{1}{\sqrt{2}}$$

218. $I = \int_0^\infty \int_0^\infty e^{-x^2(1+y^2)}\, x \, dx\, dy$

~~Limits : $x = 0$, $x = \infty$~~

~~$y = 0$, $y = \infty$~~

$$I = \int_0^\infty dy \int_0^\infty e^{-x^2(1+y^2)}\, x\, dx$$

x	0	0
t	∞	∞

Let $x^2(1+y^2) = t$

$(1+y^2)\, 2x\, dx = dt$

$$I = \frac{1}{2}\int_0^\infty dy \int_0^\infty e^{-t}\, dt$$

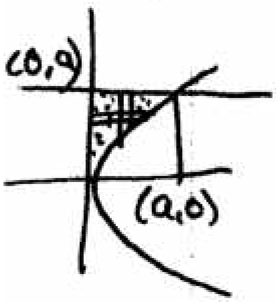
$$= \frac{1}{2}\int_0^\infty \frac{dy}{y^2+1}\left[\frac{e^{-t}}{-1}\right]_0^\infty = \frac{-1}{2}\int_0^\infty \left(\frac{0-1}{y^2+1}\right) dy$$

$$= +\frac{1}{2}\int_0^\infty \frac{dy}{y^2+1} = +\frac{1}{2}\left[\frac{1}{1}\tan^{-1}\left(\frac{y}{1}\right)\right]_0^\infty$$

$$= +\frac{1}{2}\cdot\frac{\pi}{2} = +\frac{\pi}{4}$$

99. $$I = \int_0 \int_{\sqrt{ax}} \frac{y^2\,dy\,dx}{\sqrt{y^4 - a^2x^2}}$$

Limits: $x = y^2/a$; $y = a$
$x = 0$; $x = a$



Changing order of int.
Limits: $x=0$; $x = y^2/a$;
$y=0$; $y=a$

$$I = \int_0^a \frac{y^2dy}{a} \int_0^{y^2/a} \frac{dx}{\sqrt{-x^2 + y^4/a^2}}$$

$$\cancel{= \frac{1}{a}\int_0^a y^2dy \left[\log\left(x + \sqrt{x^2 - \frac{y^4}{a^2}}\right)\right]_0^{y^2/a}}$$

$$\cancel{= \frac{1}{a}\int_0^a y^2 \left[\log\left(\frac{y^2}{a}\right) - \log\right]}$$

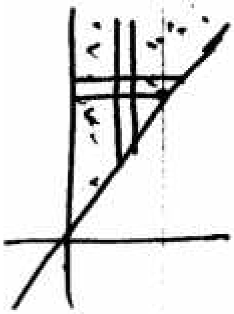
$$= \frac{1}{a}\int_0^a y^2dy \left[\sin^{-1}\left(\frac{ax}{y^2}\right)\right]_0^{y^2/a}$$

$$= \frac{\pi}{2a}\left[\frac{y^3}{3}\right]_0^a = \frac{\pi}{6a}\cdot a^3 = \frac{\pi a^2}{6}$$

23. $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Limits: $y = x$; $y = \infty$

$x = 0$; $x = \infty$



Changing order of integration

Limits: $x = 0$; $x = y$

$y = 0$; $y = \infty$

$$I = \int_0^\infty \frac{e^{-y}}{y} \int_0^y dx = \int_0^\infty \frac{e^{-y}}{\cancel{y}} [\cancel{y}]\, dy$$

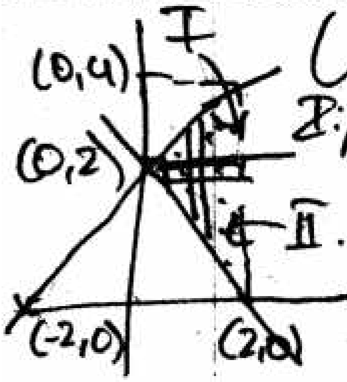
$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty = 1.$$

24. $I = \int_0^2 \int_{2-x}^{2+x} x\, dx\, dy$

Limits: $y = 2 - x \therefore x + y = 2$; $y = 2 + x$

$\therefore y - x = 2$

$x = 0$; $x = 2$



Changing order of int

I: Limits: $x = y-2$; $x = 2$

$y = 2$; $y = 4$

$$I_1 = \int_2^4 dy \int_{y-2}^{2} x\,dx$$

$$= \int_2^4 dy \left[\frac{x^2}{2}\right]_{y-2}^{2} = \frac{1}{2}\int_2^4 \left[4-(y-2)^2\right]dy$$

$$= \frac{1}{2}\left[4y - \frac{(y-2)^3}{3}\right]_2^4$$

$$= \frac{1}{2}\left[16 - \frac{8}{3} - 8 + \frac{0}{3}\right] = \frac{1}{2}\left[16 - \frac{32}{3}\right]$$

$$= 8 - \frac{16}{3}$$

$$= \frac{8}{3}$$

Region II: Limits: $x = 2-y$; $x = 2$

$y = 0$; $y = 2$

$$I_2 = \int_0^2 dy \int_{2-y}^{2} x\,dx$$

$$= \int_0^2 dy \left[\frac{x^2}{2}\right]_{2-y}^{2}$$

$$= \frac{1}{2}\int_0^2 \left[4 - (2-y)^2\right] dy$$

$$= \frac{1}{2}\left[4y - \frac{(2-y)^3}{-3}\right]_0^2$$

$$= \frac{1}{2}\left[8 - \frac{0}{-3} - \frac{8}{3}\right]$$

$$= 4 - \frac{4}{3} = \frac{8}{3}$$

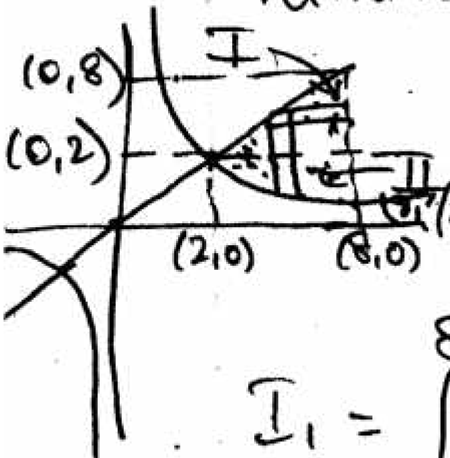
$$I = I_1 + I_2$$

$$= \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$$

23. $I = \int_{2}^{8} \int_{4/x}^{x} dy\, dx$

Limits : $xy = 4$; $y = x$

$x = 2$; $x = 8$



Changing order of int.

I: $x = y$; $x = 8$

$y = 2$; $y = 8$

$$I_1 = \int_{2}^{8} dy \int_{y}^{8} dx = \int_{2}^{8} dy \left[x\right]_{y}^{8}$$

$$= \int_{2}^{8} (8-y)\,dy = \left[8y - \frac{y^2}{2}\right]_{2}^{8}$$

$$= 64 - 32 - 16 + 2 = 18$$

II: $x = 4/y$; $x = 8$

$y = 1/2$; $y = 2$

$$I_2 = \int_{1/2}^{2} dy \int_{4/y}^{8} dx = \int_{1/2}^{2} \left(8 - \frac{4}{y}\right) dy$$

$$= \left[8y - 4\log y\right]_{1/2}^{2} = \cancel{16 - 4\log 2}$$

$$= 16 - 4\log 2 - 4 - 4\log 2$$

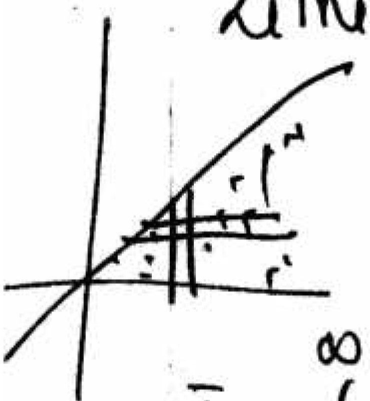
$$= 12 - 8\log 2$$

$$I = I_1 + I_2$$

$$= 18 + 12 - 8\log 2$$

$$= 30 - 8\log 2$$

23. $$I = \int_0^\infty \int_0^x x e^{-x^2/y} \, dy \, dx$$



Limits: $y = 0$; $y = x$

$x = 0$; $x = \infty$

Chang. or. of int

Limits: $x = y$; $x = \infty$

$y = 0$; $y = \infty$

$$I = \int_0^\infty dy \int_y^\infty x e^{-x^2/y} \, dx$$

Let $x^2/y = t$ $\therefore \frac{2x}{y}dx = dt$

$\therefore x\,dx = \frac{y\,dt}{2}$

x	y	∞
t	y	∞

$$I = \int_0^\infty dy \int_y^\infty e^{-t}\cdot\frac{y}{2}\,dt$$

$$= \frac{1}{2}\int_0^\infty y\,dy\left[\frac{e^{-t}}{-1}\right]_y^\infty$$

$$= \frac{1}{2}\int_0^\infty y\,dy\left[e^{-y}\right]$$

$$= \frac{1}{2}\left[\left(\frac{ye^{-y}}{-1}\right)_0^\infty - \int_0^\infty \frac{e^{-y}}{-1}\,dy\right]$$

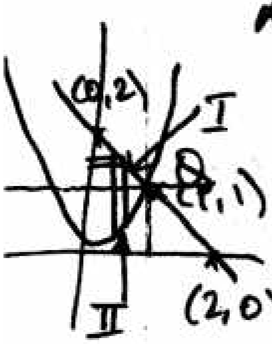
$$= \frac{1}{2}\left\{0 - \left[e^{-y}\right]_0^\infty\right)$$

$$= \frac{1}{2}(1) = \frac{1}{2}$$

27. $I = \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$

Limits: $y = x^2$; $x+y=2$

$x = 0$; $x = 1$



For Q: $x^2+x-2=0$

$x = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm 3}{2} = 1$

C.O.I.

I: Limits: $x=0$; $x = 2-y$

$y = 1$; $y = 2$

$I_1 = \int_1^2 y\,dy \int_0^{2-y} x\,dx$

$= \int_1^2 y\,dy \left[\frac{x^2}{2}\right]_0^{2-y}$

$= \int_1^2 y\left(\frac{(2-y)^2}{2}\right)dy = \cancel{\frac{1}{2}\left[\frac{y^4}{4}\right]}$

$= \cancel{\frac{1}{2}\left[4-\frac{1}{4}\right] = \frac{1}{2}\left(\frac{15}{4}\right) = \frac{15}{8}}$

$= \int_1^2 \left(2y - 2y^2 + \frac{y^3}{2}\right)dy$

$= \left[\frac{2y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{8}\right]_1^2$

$= 4 - \frac{16}{3} + 2 - 1 + \frac{2}{3} - \frac{1}{8}$

$= \frac{5-14}{3} - \frac{1}{8}$

$= \frac{120-112-3}{24}$

$= \frac{5}{24}$

II : limits: $x = 0$; $x = \sqrt{y}$

$y = 0$; $y = 1$

$$I_2 = \int_0^1 y\,dy \int_0^{\sqrt{y}} x\,dx$$

$$= \int_0^1 y\,dy \left[\frac{x^2}{2}\right]_0^{\sqrt{y}}$$

$$= \int_0^1 \frac{y^2}{2}\,dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1$$

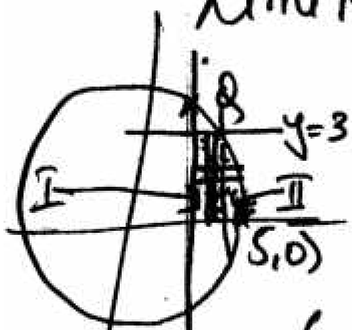
$$= \frac{1}{6}$$

$$I = I_1 + I_2 = \cancel{\frac{15}{8} + \frac{1}{6} = \frac{98}{48 \times 6} = \frac{49}{24}}$$

$$\frac{5}{24} + \frac{1}{6} = \frac{9}{24} = \frac{3}{8}$$

28. $I = \int_{0}^{3} \int_{4/3}^{\sqrt{25-y^2}} xy\,dx\,dy$

Limits: $x = \frac{4}{3}$; $x^2+y^2 = 25$



$y = 0$; $y = 3$

$8: y^2 = 25 - \frac{16}{9} = \frac{225-16}{9} = \frac{209}{9}$

C·O·I

I: Limits: $y = 0$; $y = 3$

$x = \frac{4}{3}$; $x = 4$

$$I_1 = \int_{4/3}^{4} x\,dx \int_{0}^{3} y\,dy$$

$$= \int_{4/3}^{4} x\,dx \left[\frac{y^2}{2}\right]_0^3 = \frac{9}{2}\left[\frac{x^2}{2}\right]_{4/3}^{4}$$

$$= \frac{9}{2}\left(8 - \frac{8}{9}\right)$$

II: Limits: $y = 0$; $y = \sqrt{25-x^2}$

$x = 4$; $x = 5$

$$I_2 = \int_4^5 x\,dx \int_0^{\sqrt{25-x^2}} y\,dy$$

$$= \int_4^5 x\,dx \left[\frac{y^2}{2}\right]_0^{\sqrt{25-x^2}}$$

$$= \frac{1}{2}\int_4^5 x(25-x^2)\,dx$$

$$= \frac{1}{2}\int_4^5 (25x-x^3)\,dx$$

$$= \frac{1}{2}\left[25\frac{x^2}{2} - \frac{x^4}{4}\right]_4^5$$

$$= \frac{1}{2}\left[\frac{625}{2} - \frac{625}{4} - 200 + 64\right]$$

$$= \frac{1}{2}\left[\frac{625}{4} - 136\right]$$

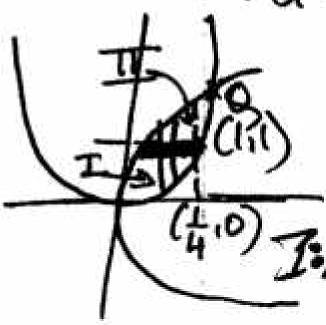
$$= \frac{1}{2}\left[\frac{625-544}{4}\right] = \frac{1}{8}(81)$$

$$I = I_1 + I_2 = 32 + \frac{81}{8} = \frac{256+81}{8} = \frac{337}{8}$$

2A $I = \int_0^{1/4} \int_{x^2}^{\sqrt{x}} dy\,dx$

Limits: $y = x^2$; $x = y^2$

$x = 0$; $x = 1/4$



For pt. Q: $y = y^4 \therefore y = 1 \therefore x = 1$

C.O.I.

I: Limits: $x = y^2$; $x = \sqrt{y}$

$y = 0$; $y = 1/16$

$$I_1 = \int_0^{1/16} dy \int_{y^2}^{\sqrt{y}} dx$$

$$= \int_0^{1/16} (\sqrt{y} - y^2)\,dy = \left[\frac{2y^{3/2}}{3} - \frac{y^3}{3}\right]_0^{1/16}$$

$$= \cancel{\left[\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{24}\right] = \frac{1}{3\sqrt{2}} - \frac{1}{24}} = \frac{2}{3}\cdot\frac{1}{16}\cdot\frac{1}{4} - \frac{1}{16^3 \cdot 3}$$

$$= \frac{1}{16}\left(\frac{1}{6} - \frac{1}{768}\right) = \frac{1 \cdot 767}{16 \cdot 6 \cdot 768}$$

II: Limits: $x = y^2$; $x = 1/4$

$y = 1/16$; $y = 1/2$

$$I_2 = \int_{1/16}^{1/2} dy \int_{y^2}^{1/4} dx = \int_{1/16}^{1/2} \left(\frac{1}{4} - y^2\right) dy$$

$$I_2 = \left[\frac{y}{4} - \frac{y^3}{3}\right]_{1/16}^{1/2} = \left(\frac{1}{8} - \frac{1}{24} - \frac{1}{64} + \frac{1}{16^3.3}\right)$$

$$I = I_1 + I_2$$

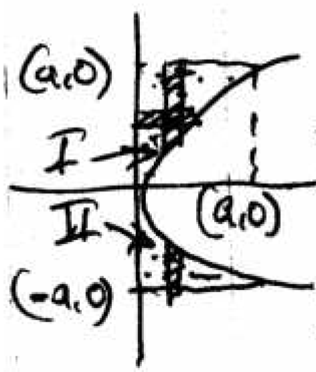
$$= \frac{1}{16.6} - \cancel{\frac{1}{16^3.3}} + \frac{1}{8} - \frac{1}{24} - \frac{1}{64} + \cancel{\frac{1}{16^3.3}}$$

$$= \frac{1}{8}\left(\frac{1}{12} + 1 - \frac{1}{3} - \frac{1}{8}\right) = \frac{1}{8}\left(\frac{-1}{24} + \frac{2}{3}\right)$$

$$= \frac{1}{8}\left(\frac{-1+16}{24}\right) = \frac{\cancel{15}\,5}{8.\cancel{24}\,8} = \frac{5}{64}$$

2b $I = \int_{-a}^{a}\int_{0}^{y^2/a} f(x,y)\,dy\,dx$

Limits: $x = 0$; $y^2 = ax$
$y = -a$; $y = a$



C.O.I.

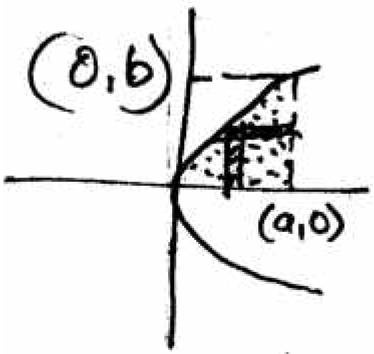
I: Limits: $y = \sqrt{ax}$; $y = a$
$x = 0$; $x = a$

II: Limits: $y = -a$; $y = -\sqrt{ax}$
$x = 0$; $x = a$

$$I = \int_0^a \int_{\sqrt{ax}}^{a} f(x,y)\,dx\,dy + \int_0^a \int_{-a}^{-\sqrt{ax}} f(x,y)\,dx\,dy$$

28 $I = \int_0^a \int_0^{b\sqrt{\frac{x}{a}}} f(x,y)\,dy\,dx$

Limits: $y = 0$; $x = \frac{ay^2}{b^2}$ $\therefore y^2 = \frac{b^2}{a}x$



$x = 0$; $x = a$

C.O.I.

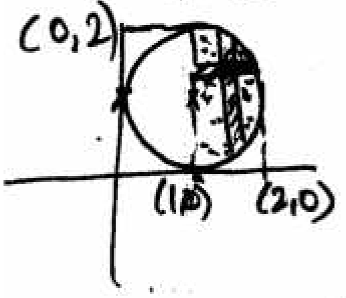
Limits: $x = \frac{ay^2}{b^2}$; $x = a$

$y = 0$; $y = b$

$$I = \int_0^b \int_{\frac{ay^2}{b^2}}^{a} f(x,y)\,dy\,dx$$

329 $I = \int_1^2 \int_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} f(x,y)\,dx\,dy$

Limits: $y = 1-\sqrt{2x-x^2}$; $y = 1+\sqrt{2x-x^2}$



$x^2+(y-1)^2-2x=0$; $x^2+(y-1)^2-2x=0$

$x^2+y^2-2y-2x+1=0$

$C\equiv(1,1)$; $r=\sqrt{1+\cancel{1}-\cancel{1}}=1$

$x=1$; $x=2$

C.O.I.

Limits: $x=1$; $\cancel{x=\sqrt{2x-(y-1)^2}}$

$x^2-2x+(y-1)^2=0$

$$x=\frac{2\pm\sqrt{4-4(y-1)^2}}{2} = 1\pm\sqrt{1-(y-1)^2}$$

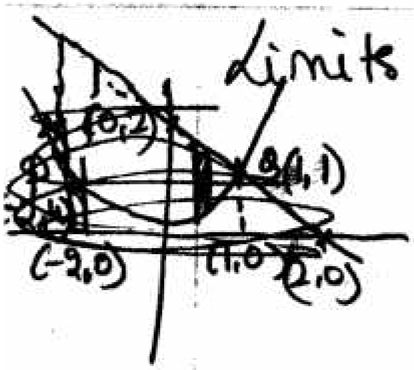
$x=1\pm\sqrt{2y-y^2}$

Limits: $x=1$; $x=1+\sqrt{2y-y^2}$

$y=0$; $y=2$.

$$I=\int_0^2\int_1^{1+\sqrt{2y-y^2}} f(x,y)\,dy\,dx$$

39 $$I=\int_{-2}^{1}\int_{x^2}^{2-x} f(x,y)\,dy\,dx$$



Limits : $y = x^2$; $x + y = 2$

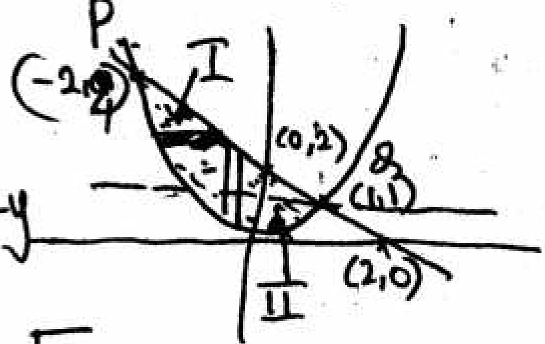
$x = -2$; $x = 1$

For Q : $x^2 + x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1 \text{ or } -2$$

For P : $(-2, 4)$

C.O.I.



I : Limits : $x = -\sqrt{y}$; $x = 2 - y$

$y = 1$; $y = 4$

II : Limits : $x = -\sqrt{y}$; $x = \sqrt{y}$

$y = 0$; $y = 1$

$$I = \int_1^4 \int_{-\sqrt{y}}^{2-y} f(x,y)\,dy\,dx + \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dy\,dx$$

3b) $$I = \int_0^b dy \int_{-\sqrt{b^2-y^2}}^{a\sqrt{1-\frac{y^2}{b^2}}} f(x,y)\,dx$$

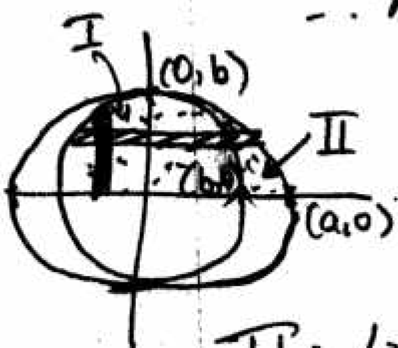
Limits: $x = -\sqrt{b^2-y^2}$; $x = a\sqrt{1-\frac{y^2}{b^2}}$

$\therefore x^2+y^2=b^2$; $x^2b^2+a^2y^2=a^2b^2$

$\therefore x^2+y^2=b^2$; $\therefore x^2/a^2+y^2/b^2=1$

$y=0$; $y=b$

C.O.I.



I : Limits : $y=0$; $y=\sqrt{b^2-x^2}$

$x=-b$; $x=b$

II : Limits : $y=\sqrt{b^2-x^2}$; $y=b\sqrt{1-\frac{x^2}{a^2}}$

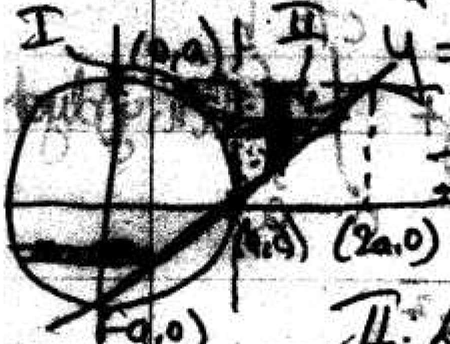
$x=b$; $x=a$

$$I=\int_{-b}^{b}\int_{0}^{\sqrt{b^2-x^2}} f(x,y)\,dy\,dx+\int_{b}^{a}\int_{\sqrt{b^2-x^2}}^{b\sqrt{1-\frac{x^2}{a^2}}} f(x,y)\,dy\,dx$$

33. $I=\int_0^a dy\int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\,dy\,dx$

Limits: $x=\sqrt{a^2-y^2}$; $x=y+a$: II

$\therefore x^2+y^2=a^2$; $x-y=a$



$y=0$; $y=a$

C.O.I

I: Limits: $y=\sqrt{a^2-x^2}$; $y=a$

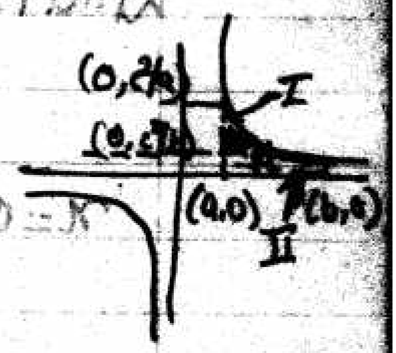
$x=0$; $x=a$

II: Limits: $y=x-a$; $y=a$

$x=a$; $x=2a$

$$I=\int_a^{2a}\int_{x-a}^{a} f(x,y)\,dy\,dx + \int_0^a\int_{\sqrt{a^2-x^2}}^{a} f(x,y)\,dy\,dx$$

38. $$I=\int_a^b\int_0^{c^2/x} f(x,y)\,dy\,dx$$



Limits: $y=0$; $xy=c^2$

$x=a$; $x=b$

C.O.I.

I: Limits: $x=a$; $x=c^2/y$

$y=\frac{c^2}{b}$; $y=\frac{c^2}{a}$

II: Limits: $x=a$; $x=b$

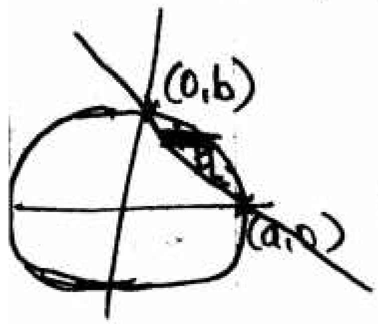
$y=0$; $y=c^2/b$

$$I = \int_{c^2/b}^{c^2/a}\int_{a}^{c^2/y} f(x,y)\,dx\,dy + \int_{0}^{c^2/b}\int_{a}^{b} f(x,y)\,dy\,dx$$

35. $I = \int_0^a \int_{\frac{b}{a}(a-x)}^{\frac{b}{a}\sqrt{a^2-x^2}} dx\,dy$ —(I)

Limits : $ay + bx = ab$; $x^2/a^2 + y^2/b^2 = 1$

$x = 0$; $x = a$



C.O.I.

Limits : $x = \frac{a}{b}(b-y)$; $x = \frac{a}{b}\sqrt{b^2-y^2}$

$y = 0$; $y = b$

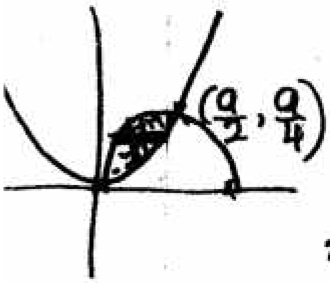
$\therefore I = \int_0^b \int_{\frac{a}{b}(b-y)}^{\frac{a}{b}\sqrt{b^2-y^2}} dx\,dy$ —(II)

$\therefore$ (I) = (II)

Represent area bet. the curves

38 $I = \int_0^{a/2} \int_{x^2/a}^{x - \frac{x^2}{a}} f(x,y)\,dx\,dy$

Limits: $x^2 = ay$; $y = x - \frac{x^2}{a}$



$x = 0$; $x = a/2$.

C.O.I.

~~Limits:~~ $x^2 - ax + ay = 0$

$\therefore x = \frac{a \pm \sqrt{a^2 - 4ay}}{2} = \frac{a}{2} - \frac{\sqrt{a^2 - 4ay}}{2}$

Limits: $x = \frac{a - \sqrt{a^2 - 4ay}}{2}$; $x = \sqrt{ay}$

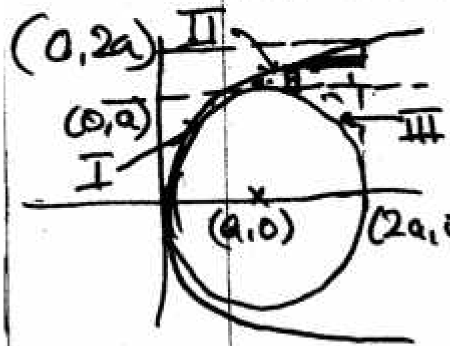
$y = 0$; $y = a/4$

$\therefore I = \int_0^{a/4} \int_{\frac{a - \sqrt{a^2 - 4ay}}{2}}^{\sqrt{ay}} f(x,y)\,dx\,dy$

39. $I=\displaystyle\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dx\,dy$

Limits: $y^2+x^2-2ax=0$; $y^2=2ax$

$x=0$; $x=2a$



$C\equiv(a,0)$; $r=\sqrt{a^2}=a$

$x=\dfrac{2a\pm\sqrt{4a^2-4y^2}}{\cancel{2}}=\dfrac{a\pm\sqrt{a^2-y^2}}{\cancel{2}}$

I: Limits: $x=\dfrac{y^2}{2a}$; $x=\dfrac{a-\sqrt{a^2-y^2}}{\cancel{2}}$

$y=0$; $y=a$

II: Limits: $x=y^2/2a$; $x=2a$

$y=a$; $y=2a$

III: Limits: $x=\dfrac{a+\sqrt{a^2-y^2}}{\cancel{2}}$; $x=2a$

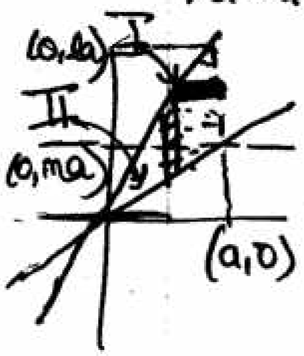
$y=0$; $y=a$

$$I=\int_0^a\int_{y^2/2a}^{\frac{a-\sqrt{a^2-y^2}}{\cancel{2}}} f(x,y)\,dx\,dy+\int_a^{2a}\int_{y^2/2a}^{2a} f(x,y)\,dx\,dy+\int_0^a\int_{\frac{a+\sqrt{a^2-y^2}}{\cancel{2}}}^{2a} f(x,y)\,dx\,dy$$

40 ~~38~~ $$I = \int_0^a \int_{mx}^{lx} f(x,y)\,dx\,dy$$

Limits: $y = mx$; $y = lx$

$x = 0$; $x = a$



I: Limits: $x = y/l$; $x = a$

$y = ma$; $y = la$

II: Limits: $x = y/l$; $x = y/m$

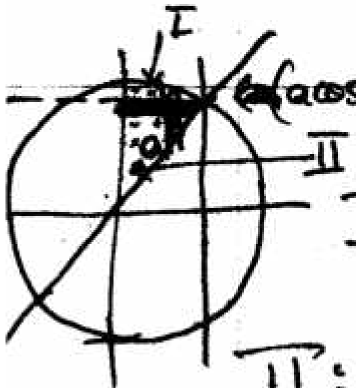
$y = 0$; $y = ma$

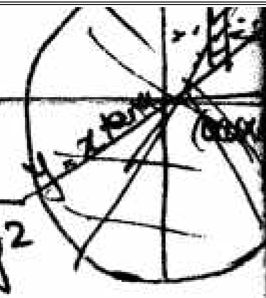
$$I = \int_{ma}^{la} \int_{y/l}^{a} f(x,y)\,dx\,dy + \int_0^{ma} \int_{y/l}^{y/m} f(x,y)\,dx\,dy$$

41/39 $$I = \int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$$

Limits: $y = x\tan\alpha$; $y^2 + x^2 = a^2$

$x = 0$; $x = a\cos\alpha$.





I : Limits : $x = 0$; $x = \sqrt{a^2 - y^2}$

$y = a\sin\alpha$; $y = a$

II : Limits : $x = 0$; $x = y/\tan\alpha$

$y = 0$; $y = a\sin\alpha$

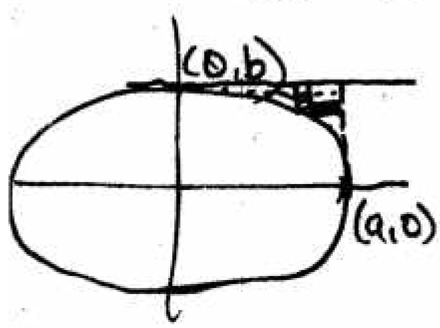
$$I = \int_{a\sin\alpha}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy + \int_{0}^{a\sin\alpha}\int_{0}^{y/\tan\alpha} f(x,y)\,dx\,dy$$

10 $\quad I = \displaystyle\int_{0}^{a}\int_{\frac{b}{a}\sqrt{a^2-x^2}}^{b} f(x,y)\,dx\,dy$

Limits : $x^2/a^2 + y^2/b^2 = 1$; $y = b$

$x = 0$; $x = a$

C.O.I :



Limits : $x = \frac{a}{b}\sqrt{b^2 - y^2}$; $x = a$

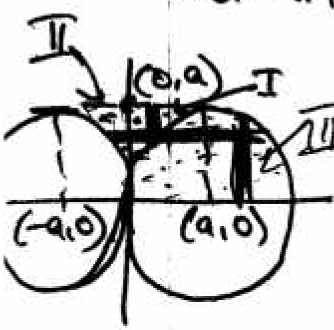
$y = 0$; $y = b$

$$I = \int_0^b \int_{\frac{a}{b}\sqrt{b^2-y^2}}^{a} f(x,y)\,dx\,dy$$

4) $$I = \int_0^a \int_{-a+\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$$

Limits: $(x+a)^2+y^2-a^2=0$; $(x-a)^2+y^2=a^2$

$C\equiv(-a,0)$; $r=a$; $C\equiv(a,0)$; $r=a$



$y=0$; $y=a$

I: Limits: $y=\sqrt{a^2-(x-a)^2}$; $y=a$

$\therefore y=\sqrt{2ax-x^2}$; $y=a$

$x=0$; $x=a$

II: Limits: $y=\sqrt{a^2-(x+a)^2}$; $y=a$

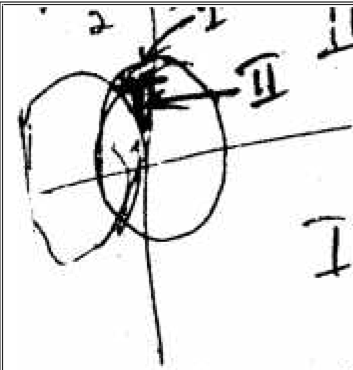
$y=\sqrt{-2ax-x^2}$; $y=a$

$x=-a$; $x=0$

$$I=\int_0^a\int_{\sqrt{2ax-x^2}}^{a} f(x,y)\,dx\,dy+\int_{-a}^{0}\int_{\sqrt{-2ax-x^2}}^{a} f(x,y)\,dx\,dy$$

III: $y=0$; $y=\sqrt{2ax-x^2}$

$x=0$; $x=2a$



$x = -a/2 \; ; \; x = 0$

$$I = \int_{-a}^{-a/2}\int_{\sqrt{-2ax-x^2}}^{a} f(x,y)\,dx\,dy + \int_{-a/2}^{0}\int_{\sqrt{a^2-x^2}}^{a} f(x,y)\,dx\,dy$$

$D: \quad x = -\sqrt{a^2-y^2}, \quad x = 0$

$\quad y = \frac{a\sqrt{3}}{2}, \quad y = a$

$D: \quad x = -a+\sqrt{a^2-y^2}, \quad x = 0$

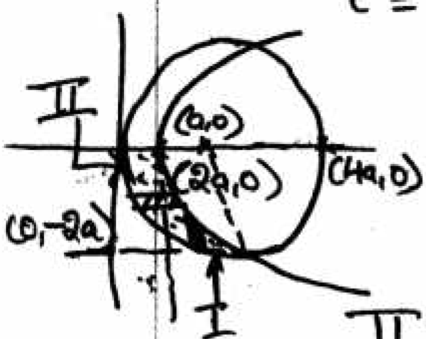
$\quad y = 0, \quad y = \frac{a\sqrt{3}}{2}$

45 $I = \int_{-2a}^{0} \int_{2a-\sqrt{4a^2-y^2}}^{a+\frac{y^2}{4a}} f(x,y)\,dx\,dy$

Limits: $(x-2a)^2 + y^2 = 4a^2$; $y^2 = 4a(x-a)$

$C \equiv (2a,0)$; $r = 2a$

$y = -2a$; $y = 0$



I: Limits: $y = -\sqrt{4a^2-(x-2a)^2}$; $y = -\sqrt{4a(x-a)}$

$y = -\sqrt{4ax - x^2}$; $y = -\sqrt{4a(x-a)}$

$x = a$; $x = 2a$

II: Limits: $y = -\sqrt{4ax - x^2}$; $y = 0$

$x = 0$; $x = a$

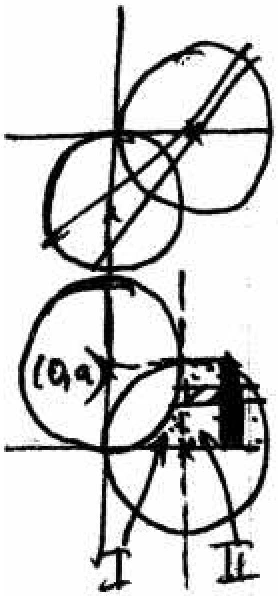
$I = \int_{a}^{2a} \int_{-\sqrt{4ax-x^2}}^{-\sqrt{4a(x-a)}} f(x,y)\,dx\,dy + \int_{0}^{a} \int_{-\sqrt{4ax-x^2}}^{0} f(x,y)\,dx\,dy.$

46. $I = \int_0^a \int_{\sqrt{2ay-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx\,dy.$

Limits: $x^2+y^2-2ay=0$; $(x-a)^2+y^2=a^2$

$C \equiv (a,a)$; $r=a$; $C \equiv (a,0)$, $r=a$

$y=0$; $y=a$



I: Limits: $y=0$; $y=a-\sqrt{a^2-x^2}$

$x=0$; $x=a$

II: Limits: $y=0$; $y=\sqrt{a^2-(x-a)^2}$

$y=0$; $y=\sqrt{2ax-x^2}$

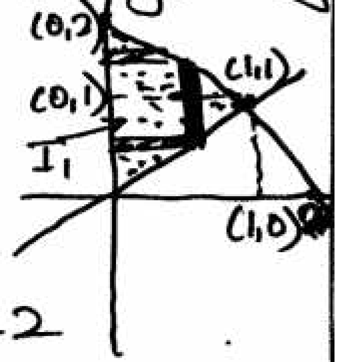
$x=a$; $x=2a$

$$I = \int_0^a \int_0^{a-\sqrt{a^2-x^2}} f(x,y)\,dx\,dy + \int_a^{2a} \int_0^{\sqrt{2ax-x^2}} f(x,y)\,dx\,dy.$$

47

45 i) $I = \int_0^1 \int_0^y (x^2+y^2)\,dx\,dy + \int_1^2 \int_0^{2-y} (x^2+y^2)\,dx\,dy$

$= I_1 + I_2$



Limits of I_1: $x=0$; $x=y$
$y=0$; $y=1$

Limits of I_2: $x=0$; $x+y=2$
$y=1$; $y=2$

C.O.I.

Limits of I: $y=x$; $y=2-\cancel{y}x$
$x=0$; $x=1$

$$I = \int_0^1 \int_x^{2-\cancel{y}x} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 dx \left[x^2y + \frac{y^3}{3}\right]_x^{2-x}$$

$$= \int_0^1 \left[x^2(2-x) + \frac{(2-x)^3}{3} - x^3 - \frac{x^3}{3}\right] dx$$

$$I = \int_0^1 \left[2x^2 - 2x^3 - \frac{x^3}{3} + \frac{(2-x)^3}{3} \right] dx$$

$$= \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3} \right] dx$$

$$= \left[2\frac{x^3}{3} - \frac{7x^4}{12} + \frac{(2-x)^4}{-12} \right]_0^1$$

$$= \frac{2}{3} - \frac{7}{12} - \frac{1}{12} + \frac{2^4}{12}$$

$$= \frac{\cancel{2}}{\cancel{3}} - \frac{\cancel{8}}{\cancel{12}} + \frac{\cancel{16}\,4}{\cancel{12}\,3} = \frac{4}{3}$$

48(iii) $I = \int_0^{a/\sqrt{2}} \int_0^{x} x\,dx\,dy + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} x\,dx\,dy$

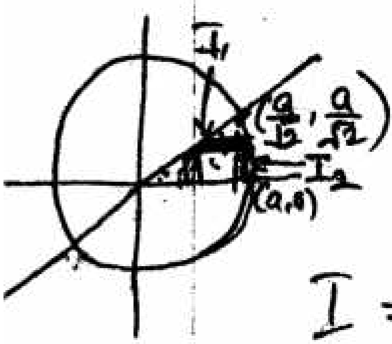
$I = I_1 + I_2$

Limits of I_1 : $y = 0$; $y = x$

$x = 0$; $x = a/\sqrt{2}$

Limits of I_2 : $y = 0$; $y^2 + x^2 = a^2$

$x = a/\sqrt{2}$; $x = a$



Limits of I : $x = y$; $x = \sqrt{a^2-y^2}$

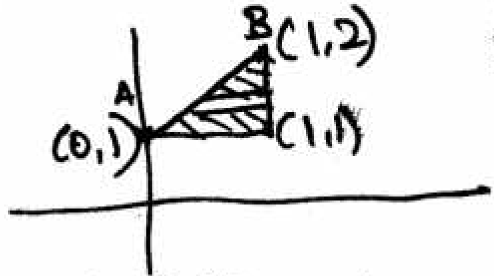
$y = 0$; $y = a/\sqrt{2}$

$$I = \int_0^{a/\sqrt{2}} dy \int_y^{\sqrt{a^2-y^2}} x\,dx = \frac{1}{2}\int_0^{a/\sqrt{2}} \left[a^2 - y^2 - y^2\right] dy$$

$$I = \frac{1}{2}\left[a^2 y - \frac{2y^3}{3}\right]_0^{a/\sqrt{2}} = \frac{1}{2}\left(\frac{a^3}{\sqrt{2}} - \frac{a^3}{3\sqrt{2}}\right)$$

$$= \frac{a^3}{2\sqrt{2}}\left(\frac{2}{3}\right) = \frac{a^3}{3\sqrt{2}}$$

49. $\iint (x^2-y^2)\,dx\,dy$



$$I=\int_1^2\int_{y-1}^{1}(x^2-y^2)\,dx\,dy$$

Eq. of AB $\Rightarrow y-1=x$

$$=\int_1^2 dy\int_{y-1}^{1}(x^2-y^2)\,dx$$

$$=\int_1^2 dy\left[\frac{x^3}{3}-xy^2\right]_{y-1}^{1}$$

$$=\int_1^2 dy\left[\frac{1}{3}-y^2-\frac{(y-1)^3}{3}+y^2(y-1)\right]$$

$$=\int_1^2\left(\frac{1}{3}-y^2-\frac{(y-1)^3}{3}+y^3-y^2\right)dy$$

$$=\int_1^2\left(\frac{1}{3}-2y^2-\frac{(y-1)^3}{3}+y^3\right)dy$$

$$=\left[\frac{y}{3}-\frac{2}{3}y^3-\frac{(y-1)^4}{12}+\frac{y^4}{4}\right]_1^2$$

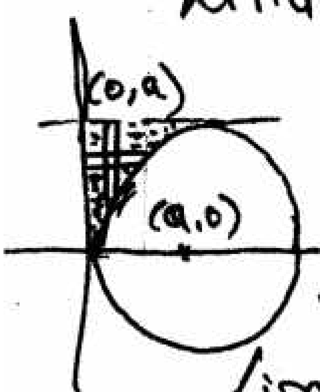
$$= \left(\frac{2}{3} - \frac{16}{3} - \frac{1}{12} + 4 - \frac{1}{3} + \frac{2}{3} - \frac{1}{4} \right)$$

$$= \left(-\frac{13}{3} - \frac{1}{3} + 4 \right) = -\frac{14}{3} + 4 = -\frac{2}{3}$$

58 $\displaystyle\int_0^a dy \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx$

Limits: $x = 0$; $x = a - \sqrt{a^2 - y^2}$

$\therefore (x-a)^2 + y^2 = a^2$



$y = 0$; $y = a$

Changing the order of integration:

Limits: $y = \sqrt{2ax - x^2}$; $y = a$

$x = 0$; $x = a$

$$I = \int_0^a dx \left[\frac{x\log(x+a)}{(x-a)^2} \right] \int_{\sqrt{2ax-x^2}}^{a} y\,dy$$

$$I = \int_0^a \frac{x \log(x+a)}{(x-a)^2} dx \left[\frac{y^2}{2} \right]_0^{\sqrt{2ax-x^2}}$$

$$= \int_0^a \frac{x \log(x+a)}{(x-a)^2} dx \left[\frac{a^2}{2} - \frac{2ax+x^2}{2} \right]$$

$$= +\frac{1}{2} \int_0^a \frac{x \log(x+a)}{\cancel{(x-a)^2}} \cancel{(x-a)^2} dx$$

$$= \frac{1}{2} \left\{ \left[\log(x+a) \cdot \frac{x^2}{2} \right]_0^a - \frac{1}{2} \int_0^a \frac{x^2 dx}{x+a} \right\}$$

$$= \frac{1}{2} \left[(\log 2a) \frac{a^2}{2} - \frac{1}{2} \left(\int_0^a \frac{x^2-a^2}{x+a} + \frac{a^2 dx}{x+a} \right) \right.$$

$$= \frac{1}{2} \left[\frac{a^2}{2} \log 2a - \frac{1}{2} \left(\int_0^a (x-a) dx + a^2 \int_0^a \frac{dx}{x+a} \right) \right.$$

$$= \frac{1}{4} \left[a^2 \log 2a - \left[\frac{x^2}{2} - ax + a^2 \log(x+a) \right]_0^a \right.$$

$$= \frac{1}{4} \left[a^2 \log 2a - \left(\frac{a^2}{2} - a^2 + a^2 \log 2a \right) - a^2 \log a \right)$$

$$= \frac{1}{4} \left[\cancel{a^2 \log 2a} - \frac{a^2}{2} + a^2 - \cancel{a^2 \log 2a} + a^2 \log a \right]$$

$$= \frac{1}{4}\left[\frac{a^2}{2} + a^2\log a\right] = \frac{a^2}{4}\left[\frac{1+2\log a}{2}\right]$$

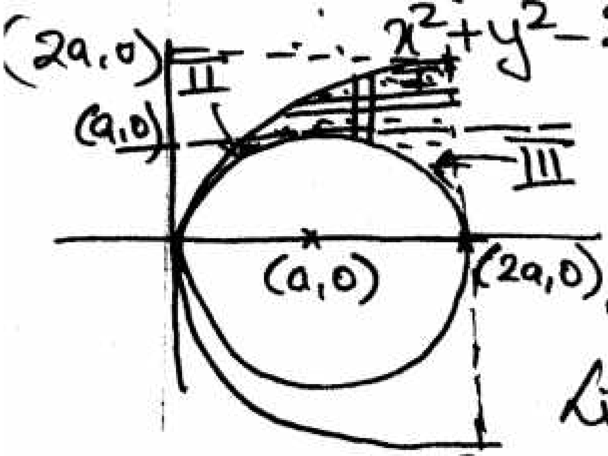
$$= \frac{a^2}{8}(2\log a + 1)$$

5) $I = \int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dx\,dy.$

Limits: $y = \sqrt{2ax - x^2}$; $y = \sqrt{2ax}$

$\therefore x^2 + y^2 - 2ax = 0$; $y^2 = 2ax$

$x = 0$; $x = 2a$



~~Reg~~ Changing the order of integration

Region I:

Limits: $x = y^2/2a$; $x = 2a$

$y = a$; $y = 2a$

$$I_1 = \int_a^{2a} \int_{y^2/2a}^{2a} f(x,y)\,dx\,dy$$

Region II:

Limits: $x = \frac{y^2}{2a}$; $x = \frac{2a \pm \sqrt{4a^2 - 4y^2}}{2}$

$x = \frac{y^2}{2a}$; $x = a - \sqrt{a^2 - y^2}$ {-ve sign ∵ ① region has x-coord. less than a}

$y = 0$; $y = a$

$$I_2 = \int_0^a \int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$$

Region III:

Limits: $x = a + \sqrt{a^2 - y^2}$ {+ve sign ∵ ① region has x-coord. greater than a}

to $x = 2a$

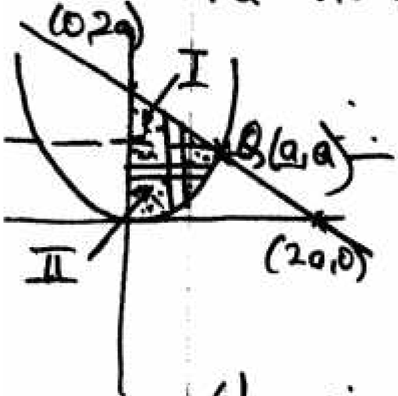
$y = 0$; $y = a$

$$I_3 = \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx\,dy$$

$$\therefore I = I_1 + I_2 + I_3$$

5. $I = \int_0^a \int_{x^2/a}^{2a-x} xy \, dx \, dy$

Limits: $y = \frac{x^2}{a}$; $y = 2a - x$



$\therefore$ $x^2 = ay$; $x + y = 2a$

$x = 0$; $x = a$

For Q: $x + \frac{x^2}{a} = 2a$ $\therefore x^2 + ax - 2a^2 = 0$

$x = \frac{-a \pm \sqrt{a^2 + 8a^2}}{2}$

$x = \frac{-a \pm 3a}{2} = a$

$\therefore y = a$

Changing order of integration:

Region I:

Limits: $x = 0$; $x = 2a - y$

$y = a$; $y = 2a$

$I_1 = \int_a^{2a} \int_0^{2a-y} xy \, dx \, dy$

Region II:

Limits: $x = 0$; $x = \sqrt{ay}$

$y = 0$; $y = a$

$$I_2 = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy$$

$$I = I_1 + I_2$$

$$I = \int_0^{2a} y\,dy \int_0^{2a-y} x\,dx + \int_0^a y\,dy \int_0^{\sqrt{ay}} x\,dx$$

$$= \int_a^{2a} y\,dy \left[\frac{x^2}{2}\right]_0^{2a-y} + \int_0^a y\,dy \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}}$$

$$= \frac{1}{2}\int_a^{2a} y(2a-y)^2\,dy + \frac{1}{2}\int_0^a ay^2\,dy$$

$$= \frac{1}{2}\int_a^{2a} (4a^2y - 4ay^2 + y^3)\,dy + \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{\cancel{4}^{2}a^2y^2}{\cancel{2}} - \frac{4ay^3}{3} + \frac{y^4}{4}\right]_a^{2a} + \frac{a^4}{6}$$

$$= \frac{1}{2}\left(8a^4 - \frac{32a^4}{3} + 4a^4 - 2a^4 + \frac{4a^4}{3} - \frac{a^4}{4}\right) + \frac{a^4}{6}$$

$$= \frac{1}{2}\left(\frac{4a^4}{3} + \frac{4a^4}{3} - \frac{9a^4}{4}\right) + \frac{a^4}{6} = \cancel{\frac{2a^4}{3} + \frac{a^4}{6} = \frac{5a^4}{6}}$$

$$= \frac{1}{2}\left(\frac{8a^4}{3} - \frac{9a^4}{4}\right) + \frac{a^4}{6}$$

$$= \frac{1}{2}\left(\frac{5a^4}{12}\right) + \frac{a^4}{6} = \frac{5a^4}{24} + \frac{a^4}{6} = \frac{9a^4}{24}$$

$$= \frac{3a^4}{8}$$

Type II: Polar Co-ordinates:-

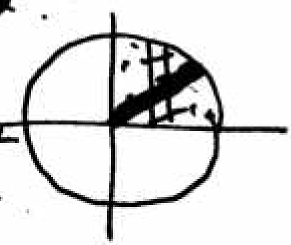
Convert cartesian to polar co-ordinates with the presence of (x^2+y^2) terms in the integrand.

Q1. $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sin\left\{\frac{\pi}{a^2}(a^2-x^2-y^2)\right\} dx\,dy$

Converting to polar co-ord:

$x = r\cos\theta,\ y = r\sin\theta$

$\therefore dx\,dy = r\,dr\,d\theta \quad \therefore x^2+y^2=r^2$



Limits: $y=0 ;\ y^2+x^2=a^2$

$x=0 ;\ x=a$

Limits: $r=0 ;\ r=a$

$\theta=0 ;\ \theta=\pi/2$

$I = \int_0^{\pi/2} \int_0^a \sin\left\{\frac{\pi}{a^2}(a^2-r^2)\right\} r\,dr\,d\theta$

$= \int_0^{\pi/2} d\theta \int_0^a r.\sin\left\{\frac{\pi}{a^2}(a^2-r^2)\right\} dr$

Let $\frac{\pi}{a^2}(a^2-r^2) = u$

$\therefore -\frac{\pi}{a^2}(2r\,dr) = du$

$\therefore r\,dr = -\frac{a^2}{2\pi}du$

r	0	a
u	π	0

$$I = \int_0^{\pi/2} d\theta \int_{\pi}^{0} \sin u\left(-\frac{a^2}{2\pi}\right)du$$

$$= -\frac{a^2}{2\pi}\int_0^{\pi/2}\left[-\cos u\right]_{\pi}^{0} d\theta$$

$$= -\frac{a^2}{2\pi}\int_0^{\pi/2}(-1-1)\,d\theta = -\frac{a^2}{2\pi}\left[-2\cdot\frac{\pi}{2}\right]$$

$$= a^2/2$$

Eg 2. $I = \int_0^{a/\sqrt{2}}\int_y^{\sqrt{a^2-y^2}} \log(x^2+y^2)\,dx\,dy$

Limits: $x = y$; $x^2+y^2 = a^2$

$y = 0$; $y = a/\sqrt{2}$

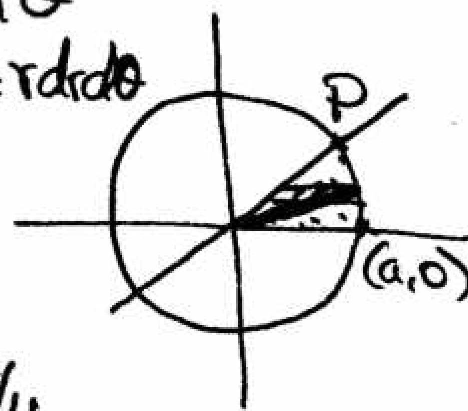
Changing to polar co-ord:

$x = r\cos\theta$; $y = r\sin\theta$

$\therefore\ x^2+y^2 = r^2$; $dx\,dy = r\,dr\,d\theta$

Limits:

$r = 0$, $r = a$



$P \equiv (a/\sqrt{2}, a/\sqrt{2})$

$\frac{a}{\sqrt{2}} = a\cos\theta$; $\theta = \pi/4$

$\theta = 0$; $\theta = \pi/4$

$$I = \int_0^{\pi/4}\int_0^{a} \log r^2 (r\,dr\,d\theta)$$

$$= 2\int_0^{\pi/4} d\theta \int_0^{a} \underset{v}{r}\,\underset{u}{\log r}\,dr$$

$$= 2\int_0^{\pi/4} d\theta \left[\log r \cdot \frac{r^2}{2}\Big|_0^a - \int_0^a \frac{1}{r}\cdot\frac{r^2}{2}\,dr \right]$$

$$= 2\int_0^{\pi/4} d\theta \left[\frac{a^2}{2}\log a - \frac{a^2}{4} \right]$$

$$= 2\left[\frac{a^2}{2}\log a - \frac{a^2}{4}\right]\left(\frac{\pi}{4}\right)$$

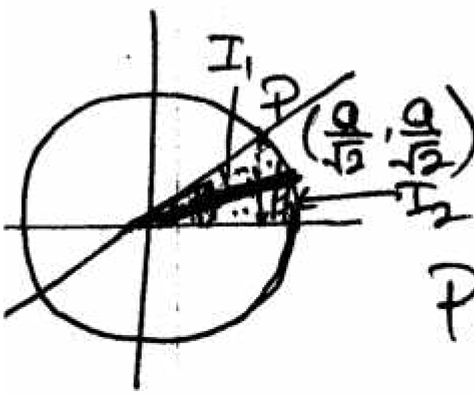
$$I = \frac{\pi a^2}{8}(2 \log a - 1)$$

Eg 3. $$I = \int_0^{a/\sqrt{2}} \int_0^{x} \cos[k(x^2+y^2)]\,dx\,dy + \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} \cos[k(x^2+y^2)]\,dx\,dy$$

$I = I_1 + I_2$

Limits of I_1: $y=0$; $y=x$
$x=0$; $x=a/\sqrt{2}$

Limits of I_2: $y=0$; $x^2+y^2=a^2$
$x=a/\sqrt{2}$; $x=a$



Changing to polar co-ord.
$x = r\cos\theta$; $y = r\sin\theta$
$\therefore x^2+y^2=r^2$; $dx\,dy = r\,dr\,d\theta$

$P \equiv \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$

$x = r\cos\theta$; $a/\sqrt{2} = a\cos\theta$ $\therefore \theta = \pi/4$.

Limits :

$r = 0$; $r = a$

$\theta = 0$; $\theta = \pi/4$

$$I = \int_0^{\pi/4} d\theta \int_0^a \cos[kr^2]\, r\,dr$$

Let $kr^2 = u$

$\therefore r\,dr = \frac{du}{2k}$

r	0	a
u	0	ka^2

$$I = \int_0^{\pi/4} d\theta \int_0^{ka^2} \cos u \left(\frac{du}{2k}\right)$$

$$I = \frac{1}{2k}\int_0^{\pi/4} d\theta \left[\sin u\right]_0^{ka^2}$$

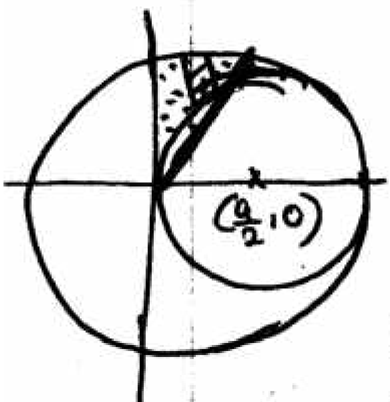
$$= \frac{1}{2k}\sin ka^2 \left[\frac{\pi}{4}\right]$$

$$= \frac{\pi}{8k}\sin ka^2$$

eg4. $$I = \int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy}{x^2+y^2} e^{-(x^2+y^2)}\, dx\,dy$$

Limits: $y^2+x^2-ax=0$; $y^2+x^2=a^2$

$x=0$; $x=a$.



Changing to polar co-ordi

$x=r\cos\theta$, $y=r\sin\theta$

$dx\,dy = r\,dr\,d\theta$ ∴ $x^2+y^2=r^2$

Limits: $r=a\cos\theta$; $r=a$

$\theta=0$; $\theta=\pi/2$.

{∵ Strip is from inner ⊙ to outer ⊙}

$$I = \int_0^{\pi/2} d\theta \int_{a\cos\theta}^{a} \frac{r\cos\theta \, r\sin\theta}{r^2} \cdot e^{-r^2} \cdot r\,dr$$

$$= -\frac{1}{2}\int_0^{\pi/2} d\theta \int_a^{a\cos\theta} e^{-r^2}(-2r\,dr)\sin\theta\cos\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta \left[e^{-r^2}\right]_a^{a\cos\theta} \sin\theta\cos\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(e^{-a^2\cos^2\theta} - e^{-a^2}\right) d\theta \, \sin\theta\cos\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2} e^{-a^2\cos^2\theta}\left(2a^2\cos\theta\sin\theta\right)\frac{1}{2a^2}\,d\theta$$

$$- \frac{1}{2}\int_0^{\pi/2} e^{-a^2}\sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{4a^2}\left[e^{-a^2\cos^2\theta}\right]_0^{\pi/2} - \frac{e^{-a^2}}{2}\cdot\frac{1}{2}\beta(2,2)$$

$$= \frac{1}{4a^2}\left[1 - e^{-a^2}\right] - \frac{e^{-a^2}}{4}(1)$$

Eg5. $I = \iint_R \sqrt{\dfrac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}}\, dx\,dy$

where R is region bounded by ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Changing to polar co-ord.

Let $x = ar\cos\theta$, $y = br\sin\theta$

$\therefore dx\,dy = ab\,r\,dr\,d\theta$

Limits: $r = 0$; $r = 1$

$\theta = 0$; $\theta = 2\pi$

$$I = \int_0^{2\pi}\int_0^1 \sqrt{\frac{a^2b^2 - b^2a^2r^2\cos^2\theta - a^2b^2r^2\sin^2\theta}{a^2b^2 + b^2a^2r^2\cos^2\theta + a^2b^2r^2\sin^2\theta}}$$

$$= ab\int_0^{2\pi} d\theta \int_0^1 r\sqrt{\frac{1-r^2}{1+r^2}}\,dr \times \frac{\sqrt{1-r^2}}{\sqrt{1-r^2}}$$ {Rational

$$= ab\int_0^{2\pi} d\theta \int_0^1 \frac{r(1-r^2)}{\sqrt{1-r^4}}\,dr$$

Let $r^2 = \sin\phi$

$\therefore\ r\,dr = \dfrac{\cos\phi}{2}\,d\phi$

r	0	1
ϕ	0	$\pi/2$

$$I = ab\int_0^{2\pi} d\theta \cdot \int_0^{\pi/2} \frac{\cancel{\cos\phi}}{2}\,\frac{(1-\sin\phi)}{\cancel{\cos\phi}}\,d\phi$$

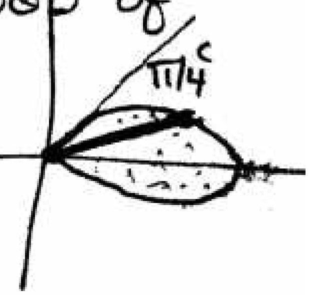
$$= \frac{ab}{2}\int_0^{2\pi} d\theta\,\Big[\phi + \cos\phi\Big]_0^{\pi/2}$$

$$= \frac{ab}{2}\int_0^{2\pi}\left(\frac{\pi}{2} - 1\right)d\phi$$

$$= \frac{ab}{\cancel{2}}\left(\frac{\pi}{2}-1\right)\cdot\cancel{2}\pi = ab\pi\left(\frac{\pi}{2}-1\right)$$

abrdrdθ Eg6 $I = \displaystyle\iint \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$ over one loop of

$r^2 = a^2\cos 2\theta$



Limits: $r = 0$; $r = \sqrt{a^2\cos 2\theta}$

$\theta = 0$; $\theta = \pi/4$

$$I = \frac{2}{2}\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{2r}{\sqrt{a^2+r^2}}\,dr\,d\theta$$

$$I = \int_0^{\pi/4} \left[\frac{\sqrt{a^2+r^2}}{1/2} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\int_0^{\pi/4} \left(\sqrt{a^2+a^2\cos 2\theta} - a\right) d\theta$$

$$= 2a \int_0^{\pi/4} \left(\sqrt{1+\cos 2\theta} - 1\right) d\theta$$

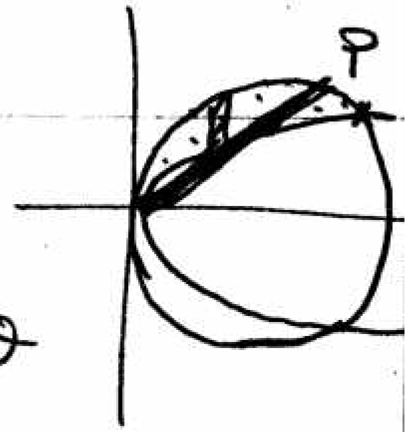
$$= 2a \int_0^{\pi/4} \left(\sqrt{2}\cdot\cos\theta - 1\right) d\theta$$

$$= 2a \left[\sqrt{2}\sin\theta - \theta \right]_0^{\pi/4}$$

$$= 2a\left[\cancel{\sqrt{2}}\cdot\frac{1}{\cancel{\sqrt{2}}} - \frac{\pi}{4}\right] = 2a\left(1-\frac{\pi}{4}\right)$$

Eg 7. $$I = \int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\, dx\, dy$$

Limits: $y^2 = 4ax$; $y^2+x^2-5ax=0$
$x = 0$; $x = a$



$P \equiv (a, 2a)$

Changing to P.C.

$x = r\cos\theta \;;\; y = r\sin\theta$

$\therefore dxdy = r\,dr\,d\theta$

$a = r\cos\theta \;;\; 2a = r\sin\theta$

$\therefore \tan\theta = 2 \;\therefore\; \theta = \tan^{-1}2.$

Eq. of Parabola: $r^2\sin^2\theta = 4ar\cos\theta$

$\therefore r = \dfrac{4a\cos\theta}{\sin^2\theta}$

Eq. of ⊙ : $r^2 = 5a\cos\theta$

Limits: $r = \dfrac{4a\cos\theta}{\sin^2\theta} \;;\; r = 5a\cos\theta$

$\theta = \tan^{-1}(2) \;;\; \theta = \pi/2$

$$I = \int_{\tan^{-1}(2)}^{\pi/2} d\theta \int_{\frac{4a\cos\theta}{\sin^2\theta}}^{5a\cos\theta} \frac{\cancel{r}}{r^{\cancel{2}}\sin^2\theta} \cdot \cancel{r}\,dr$$

$$= \int_{\tan^{-1}(2)}^{\pi/2} \left[5a\cos\theta - \frac{4a\cos\theta}{\sin^2\theta}\right] \frac{1}{\sin^2\theta}\, d\theta$$

$$I = \int_{\tan^{-1}(2)}^{\pi/2} 5a\,\frac{\cos\theta\, d\theta}{\sin^2\theta} - 4a\int_{\tan^{-1}(2)}^{\pi/2} \frac{\cos\theta\, d\theta}{\sin^4\theta}$$

$$= 5a\left[-\frac{1}{\sin\theta}\right]_{\tan^{-1}(2)}^{\pi/2} - 4a\left[\frac{1}{-3\sin^3\theta}\right]_{\tan^{-1}(2)}^{\pi/2}$$

Let $(\tan^{-1}(2)) = \theta$

$\therefore \tan\theta = 2 \quad \therefore \sin\theta = 2/\sqrt{5}$

$\therefore \sin(\tan^{-1}(2)) = 2/\sqrt{5}$.

$$I = 5a\left(-1 + \frac{\sqrt{5}}{2}\right) + \frac{4a}{3}\left(1 - \frac{5\sqrt{5}}{8}\right)$$

$$= -5a + \frac{4a}{3} + \frac{a5\sqrt{5}}{2} - \frac{a\,20\sqrt{5}}{24}$$

$$= -\frac{11a}{3} + \frac{a5\sqrt{5}}{3}$$

Eg8 S.T. $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$

By def. $\Gamma m = \int_0^{\infty} e^{-u} . u^{m-1} du$

Let $u = x^2$

$du = 2x dx$

u	0	∞
x	0	∞

$\Gamma m = 2\int_0^{\infty} e^{-x^2} (x^2)^{m-1} x dx$

$= 2\int_0^{\infty} e^{-x^2} x^{2m-2} . x^1 dx$

$\therefore \Gamma m = 2\int_0^{\infty} e^{-x^2} x^{2m-1} dx \quad - (I)$

$||^{ly}\ \Gamma n = 2\int_0^{\infty} e^{-y^2} y^{2n-1} dy$

$\Gamma m \Gamma n = 4\int_0^{\infty} e^{-x^2} x^{2m-1} dx \int_0^{\infty} e^{-y^2} y^{2n-1} dy$

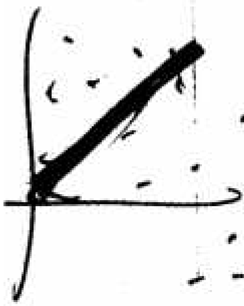
$\because$ limits are consts.

$$\Gamma m\,\Gamma n = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,x^{2m-1}\,y^{2n-1}\,dx\,dy$$

Changing to polar co-ord.

$$x = r\cos\theta,\quad y = r\sin\theta$$

$$dx\,dy = r\,dr\,d\theta$$



Limits: $r=0$; $r=\infty$

$\theta = 0$; $\theta = \pi/2$

$$\therefore \Gamma m\,\Gamma n = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r^{2m-1+2n-1}\cdot\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,r\,dr\,d\theta$$

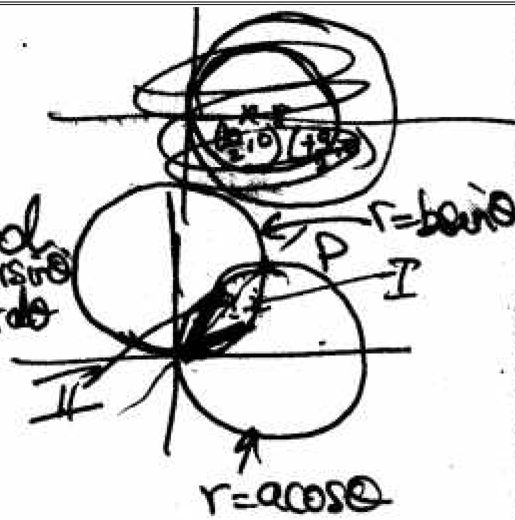
$$= 4\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta\int_0^\infty e^{-r^2}\,r^{2(m+n)-1}\,dr$$

$$= \left[2\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta\right]\left[2\int_0^\infty e^{-r^2}\,r^{2(m+n)-1}\,dr\right]$$

$$\therefore \Gamma m\,\Gamma n = \beta(m,n)\cdot\Gamma(m+n)$$

$$\therefore \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

89. $\displaystyle\iint \frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$



Changing to polar co-ord.

$x = r\cos\theta,\ y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = ax$

Eq. circle: $r = a\cos\theta$

$x^2+y^2 = by$

Eq. circle: $r = b\sin\theta$

For P: $\theta = \tan^{-1}(a/b)$

$I = I_1 + I_2$

Limits I_1: $r_1 = 0$; $r_2 = \cancel{a\cos\theta}$ $b\sin\theta$

$\theta_1 = 0$; $\theta_2 = \tan^{-1}(a/b)$

Limits I_2: $r_1 = 0$; $r_2 = a\cos\theta$

$\theta_1 = \tan^{-1}(a/b)$; $\theta_2 = \pi/2$

$$I = \int_0^{\tan^{-1}(a/b)} \int_0^{b\sin\theta} \frac{\cancel{r^4}}{\cancel{r^4}\cos^2\theta\sin^2\theta}\cdot r\,dr\,d\theta + \int_{\tan^{-1}(a/b)}^{\pi/2} \int_0^{a\cos\theta} \frac{r\,dr\,d\theta}{\sin^2\theta\cos^2\theta}$$

$$I = \int_0^{\tan^{-1}(a/b)} \frac{d\theta}{\cos^2\theta\,\sin^2\theta}\left[\frac{r^2}{2}\right]_0^{b\sin\theta} + \int_{\tan^{-1}(a/b)}^{\pi/2} \frac{d\theta}{\sin^2\theta\cos^2\theta}\left[\frac{r^2}{2}\right]_0^{a\cos\theta}$$

$$= \frac{b^2}{2}\int_0^{\tan^{-1}(a/b)} \frac{d\theta}{\cos^2\theta} + \frac{a^2}{2}\int_{\tan^{-1}(a/b)}^{\pi/2} \frac{d\theta}{\sin^2\theta}$$

$$= \frac{b^2}{2}\left[\tan\theta\right]_0^{\tan^{-1}(a/b)} - \frac{a^2}{2}\left[\cot\theta\right]_{\tan^{-1}(a/b)}^{\pi/2}$$

$$= \frac{b^2}{2}\left(\frac{a}{b}\right) - \frac{a^2}{2}\left[\frac{1}{\infty} - \frac{b}{a}\right]$$

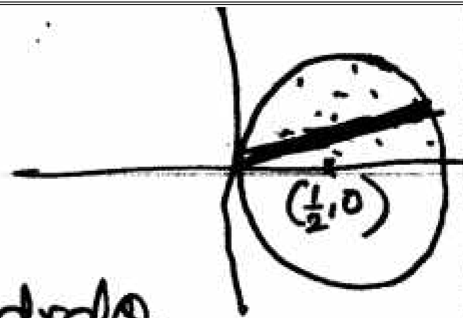
$$= \frac{ab}{2} + \frac{ab}{2} = ab.$$

12. $$I = \int_0^1 \int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2} e^{-(x^2+y^2)}\,dx\,dy$$

Limits: $y = 0$; $y^2 + x^2 - x = 0$

$x = 0$; $x = 1$

Changing to polar co-ords



$x = r\cos\theta$; $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$ & $dxdy = r\,dr\,d\theta$

Limits:

$r_1 = 0$; $r_2 = \cos\theta$

$\theta_1 = 0$; $\theta_2 = \pi/2$

$$I = \int_0^{\pi/2}\int_0^{\cos\theta} \frac{4\cancel{r^2}\cos\theta\sin\theta}{\cancel{r^2}} e^{-r^2} . r\,dr\,d\theta$$

$$= \frac{4}{-2}\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \int_0^{\cos\theta} e^{-r^2}(-2r\,dr)$$

$$= -2\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \left[e^{-r^2}\right]_0^{\cos\theta}$$

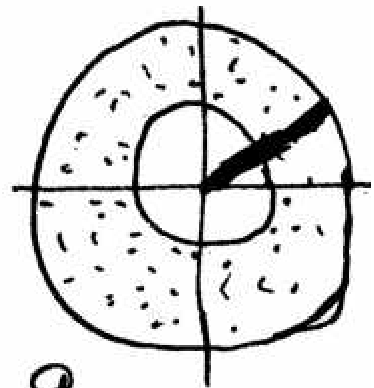
$$= -2\int_0^{\pi/2} \sin\theta\cos\theta\left(e^{-\cos^2\theta} - 1\right)d\theta$$

$$= -\frac{2}{2}\int_0^{\pi/2} e^{-\cos^2\theta}(+2\cos\theta\sin\theta)\,d\theta + \frac{2}{2}\int_0^{\pi/2} 2\sin\theta\cos\theta\,d\theta$$

$$= -\left[e^{-\cos^2\theta}\right]_0^{\pi/2} + \left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= -(1-e^{-1}) + \left(+\frac{1}{2} + \frac{1}{2}\right)$$

$$= -\cancel{1} + \frac{1}{e} + \cancel{1} = \frac{1}{e}$$



13. $I = \iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$

$x^2+y^2=4$ & $x^2+y^2=9$

Changing to polar co-ords.

$x = r\cos\theta$; $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$; $x^2+y^2=r^2$

Eq. of ⊙s : $r=2$; $r=3$

Limits : $r_1=2$; $r_2=3$

$\theta_1=0$; $\theta_2=2\pi$

$$I = \int_0^{2\pi}\int_2^3 \frac{r^4\sin^2\theta\cos^2\theta}{r^2}\,.\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\sin^2\theta\cos^2\theta\, d\theta\right] \left[\frac{r^4}{4}\right]_2^3$$

$$= \frac{1}{4}\int_0^{2\pi} (\sin 2\theta)^2\, d\theta \left(\frac{81}{4} - \frac{16}{4}\right)$$

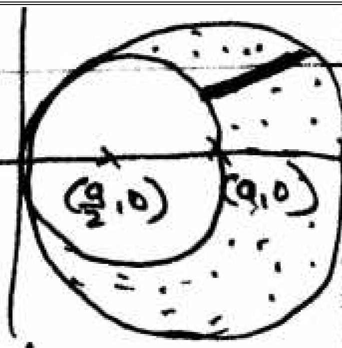
$$= \frac{65}{16}\int_0^{2\pi} \left(\frac{1-\cos 4\theta}{2}\right) d\theta$$

$$\cancel{= \frac{65}{16}\cdot\frac{1}{4}\int_0^{2\pi} \left(1 - 2\cos 4\theta + \cos^2 4\theta\right) d\theta}$$

$$= \frac{65}{16}\cdot\frac{1}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi}$$

$$= \frac{65}{16}\cdot\frac{1}{\cancel{2}}\,\cancel{2}\pi = \frac{65\pi}{16}$$

18 $I = \iint y^2\,dx\,dy$



$x^2+y^2-ax=0 \cdot x^2+y^2-2ax=0$

Changing to polar co-ord.

$x = r\cos\theta$; $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$; $x^2+y^2=r^2$

Eq. of O's : $r = a\cos\theta$; $r = 2a\cos\theta$

Limits :

$r_1 = a\cos\theta$; $r_2 = 2a\cos\theta$

$\theta_1 = -\pi/2$; $\theta_2 = \pi/2$

$$I = \int_{-\pi/2}^{\pi/2}\int_{a\cos\theta}^{2a\cos\theta} r^2\sin^2\theta\, r\,dr\,d\theta$$

$$= 2\int_{0}^{\pi/2} \sin^2\theta\,d\theta \left[\frac{r^4}{4}\right]_{a\cos\theta}^{2a\cos\theta}$$

$$= \frac{2}{4}\int_0^{\pi/2}\left(\sin^2\theta \cdot 16a^4\cos^4\theta - a^4\sin^2\theta\cos^4\theta\right)d\theta$$

$$I = \frac{15a^4}{2}\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta$$

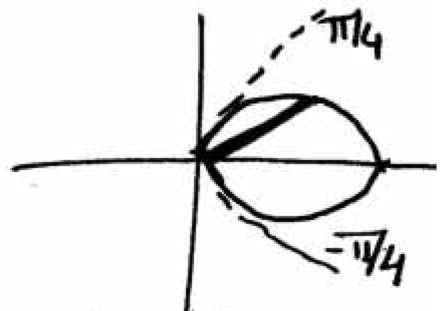
$$= \frac{15}{2}a^4 \frac{1}{2}\, \beta\left(\frac{2+1}{2}, \frac{4+1}{2}\right)$$

$$= \frac{15}{4}a^4 \beta\left(\frac{3}{2}, \frac{5}{2}\right)$$

$$= \frac{15}{4}a^4 \frac{\Gamma 3/2\, \Gamma 5/2}{\Gamma 4} = \frac{\cancel{15}^{5}}{4}a^4 \frac{\frac{1}{2}\Gamma\frac{1}{2} \cdot \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{\cancel{3}\times 2}$$

$$= \frac{15}{64} \cdot \pi a^4 .$$

13(b) $I = \iint \frac{dx\,dy}{(1+x^2+y^2)^2}$



$(x^2+y^2)^2 = x^2-y^2$

Changing to polar co-ord.

$x = r\cos\theta$; $y = r\sin\theta$

$\therefore$ $dx\,dy = r\,dr\,d\theta$; $x^2+y^2 = r^2$

Eq. of curve : $r^2 = \cos 2\theta$

Limits: $r_1 = 0$; $r_2 = \sqrt{\cos 2\theta}$

$\theta_1 = -\pi/4$; $\theta_2 = \pi/4$

$$I = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \int_0^{\sqrt{\cos 2\theta}} \frac{2r\,dr\,d\theta}{(1+r^2)^2}$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left[-\frac{1}{1+r^2} \right]_0^{\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(-\frac{1}{1+\cos 2\theta} + 1 \right) d\theta$$

$$\cancel{= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \left(\frac{\cos 2\theta}{1+\cos 2\theta} \right) d\theta}$$

$$= \frac{1}{2} \left\{ [\theta]_{-\pi/4}^{\pi/4} - \frac{1}{2} \int_{-\pi/4}^{\pi/4} \frac{d\theta}{2\cos^2\theta} \right.$$

$$= \frac{1}{2} \left(\frac{\pi}{4} + \frac{\pi}{4} \right) - \frac{1}{4} \int_{-\pi/4}^{\pi/4} \sec^2\theta\, d\theta$$

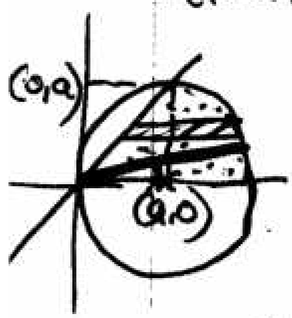
$$= \frac{\pi}{4} - \frac{1}{4}\left[\tan\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{4} - \frac{1}{4}(1+1)$$

$$= \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}$$

#14) $$I = \int_0^a \int_y^{a+\sqrt{a^2-y^2}} \frac{dx\,dy}{(4a^2+x^2+y^2)^2}$$

Limits: $x = y$; $(x-a)^2 + y^2 = a^2$

$y = 0$; $y = a$



Changing to polar co-ord.

$x = r\cos\theta$; $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$; $x^2+y^2 = r^2$

Eq. of ⊙ : $x^2+y^2-2ax = 0$

$\therefore r = 2a\cos\theta$

Limits: $r_1 = 0$, $r_2 = 2a\cos\theta$

$\theta_1 = 0$; $\theta_2 = \pi/4$

$$I = \int_0^{\pi/4} \frac{1}{2} \int_0^{2a\cos\theta} \frac{2r\,dr\,d\theta}{(4a^2+r^2)^2}$$

$$= \frac{1}{2}\int_0^{\pi/4} d\theta \left[-\frac{1}{4a^2+r^2} \right]_0^{2a\cos\theta}$$

$$= \frac{1}{2}\int_0^{\pi/4} \left(\frac{-1}{4a^2+4a^2\cos^2\theta} + \frac{1}{4a^2} \right) d\theta$$

$$= \frac{1}{8a^2}\int_0^{\pi/4} \left(\frac{-1}{1+\cos^2\theta} + 1 \right) d\theta$$

$$= \frac{1}{8a^2}\int_0^{\pi/4} d\theta - \frac{1}{8a^2}\int_0^{\pi/4} \frac{d\theta/\cos^2\theta}{\frac{1+\cos^2\theta}{\cos^2\theta}}$$

$$= \frac{1}{8a^2}\cdot\frac{\pi}{4} - \frac{1}{8a^2}\int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^2\theta+1}$$

$$= \frac{\pi}{8a^2.4} - \frac{1}{8a^2}\int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{2+\tan^2\theta}$$

Let $\tan\theta = t$

$\therefore \sec^2\theta\, d\theta = dt$

θ	0	$\pi/4$
t	0	1

$$I = \frac{\pi}{8a^2.4} - \frac{1}{8a^2}\int_0^1 \frac{dt}{2+t^2}$$

$$I = \frac{\pi}{8a^2 \cdot 4} - \frac{1}{8a^2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^1$$

$$= \frac{\pi}{8a^2 \cdot 4} - \frac{1}{8a^2}\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{8a^2}\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right)$$