I. GAMMA FUNCTIONS:-

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

Hence,

$$\int_0^\infty e^{-x} \cdot x^{n}\, dx = \Gamma n+1$$

eg(1) Find the gamma funct. of

(i) $\int_0^\infty e^{-x} \cdot x^{2}\, dx = \Gamma 2+1 = \Gamma 3$

(ii) $\int_0^\infty e^{-x} \cdot x^{-1/2}\, dx = \Gamma -1/2+1 = \Gamma 1/2$

2. Prove $\Gamma 1 = 1$

By def $\Gamma 1 = \int_0^\infty e^{-x} \cdot x^{1-1}\, dx$

$$\Gamma 1 = \int_0^\infty e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$= \left[\frac{e^{-\infty} - e^{0}}{-1}\right] = \left(\frac{0-1}{-1}\right) = 1$$

3 Prove $\Gamma(n+1) = n\Gamma n$

By def $\Gamma(n+1) = \int_0^\infty e^{-x} \cdot x^{n+1-1}\,dx$

$$= \int_0^\infty \overset{v}{e^{-x}} \cdot \overset{u}{x^n}\,dx$$

$$= \left[\frac{x^n e^{-x}}{-1}\right]_0^\infty - \int_0^\infty \frac{n x^{n-1} e^{-x}}{-1}\,dx$$

$$= \left[\frac{x^n e^{-\infty} - 0e^{0}}{-1}\right] + n\int_0^\infty e^{-x} \cdot x^{n-1}\,dx$$

$$= 0 + n\Gamma n = n\Gamma n$$

$\therefore \Gamma(n+1) = n\Gamma n$

Similarly,

$$\Gamma n = (n-1)\,\Gamma n-1$$

$$= (n-1)(n-2)\,\Gamma n-2$$

$$\therefore \Gamma n = (n-1)(n-2)(n-3) \ldots \Gamma 1$$

$$\Gamma n = (n-1)! \qquad (\because \Gamma 1 = 1)$$

Also,

$$\Gamma n+1 = n!$$

Eg. 1 $\Gamma 5 = 4! = 4 \times 3 \times 2 = 24$

$$\Gamma \tfrac{7}{2} = \left(\tfrac{5}{2}\right)! = \tfrac{5}{2} \times \tfrac{3}{2} \times \tfrac{1}{2}\,\Gamma \tfrac{1}{2} = \tfrac{15}{8}\,\Gamma \tfrac{1}{2}$$

$$= \tfrac{15}{8} \cdot \sqrt{\pi} \qquad \left(\because \Gamma \tfrac{1}{2} = \sqrt{\pi}\right)$$

Type I: ~~Problems~~ Integrals of type:-

$\int_0^\infty e^{-ax^m} \cdot x^n \, dx$ can be solved by

Substitute $ax^m = t$.

Q.1 $I = \int_0^\infty e^{-\sqrt{x}} \cdot \sqrt{x}\, dx$

Let $\sqrt{x} = t$

$x = t^2 \quad dx = 2t\,dt$

x	0	∞
t	0	∞

$= 2\int_0^\infty e^{-t} t^2 dt = 2\sqrt{3} = 2\,2! = 4$

2. $I = \int_0^\infty e^{-ax^2} dx$

Let $ax^2 = t$

$2ax\,dx = dt$

$dx = \dfrac{dt}{2ax} = \dfrac{dt}{2a\frac{\sqrt{t}}{\sqrt{a}}} = \dfrac{dt}{2\sqrt{a}\sqrt{t}}$

x	0	∞
t	0	∞

$$I = \int_0^\infty e^{-t} \cdot \frac{t^{-1/2}}{2\sqrt{a}}\, dt$$

$$= \frac{1}{2\sqrt{a}} \Gamma\left(-\tfrac{1}{2}+1\right) = \frac{1}{2\sqrt{a}} \Gamma\left(\tfrac{1}{2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\left(\because \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right)$$

3. S.T. $\int_0^\infty x e^{-ax} \sin bx\, dx = \frac{2ab}{(a^2+b^2)^2}$

$$\sin bx = I.P.\left[\cos bx + i \sin bx\right]$$

$$= I.P.\left[e^{ibx}\right]$$

$$\therefore \int_0^\infty x e^{-ax}\, I.P.\left[e^{ibx}\right] dx$$

$$= I.P.\left[\int_0^\infty x e^{-ax} \cdot e^{ibx}\, dx\right]$$

$$= I.P.\left[\int_0^\infty x e^{-x(a-ib)}\, dx\right]$$

Let $x(a-ib)=t$

$x = \dfrac{t}{a-ib}$; $dx = \dfrac{dt}{a-ib}$

x	0	∞
t	0	∞

$$I = I.P.\left[\int_0^\infty \frac{t}{a-ib}\, e^{-t}\, \frac{dt}{a-ib}\right]$$

$$= I.P.\left[\frac{1}{(a-ib)^2}\int_0^\infty e^{-t}\, t\, dt\right]$$

$$= I.P.\left[\frac{1}{(a-ib)^2}\,\Gamma 2\right]$$

$$= I.P.\left[\frac{1}{(a-ib)^2}\right]$$

$$= I.P.\left[\frac{(a+ib)^2}{(a-ib)^2(a+ib)^2}\right]$$

$$= I.P.\left[\frac{a^2-b^2+2iab}{(a^2-i^2b^2)^2}\right]$$

$$= \frac{2ab}{(a^2+b^2)^2}$$

1. S.T. $\int_0^\infty e^{-x^4}.dx = \frac{1}{4}\left|\frac{1}{4}\right.$

Let $x^4 = t \quad \therefore x = t^{1/4}$

$\therefore 4x^3 dx = dt$

$dx = \frac{dt}{4x^3} = \frac{dt}{4t^{3/4}}$

$$I = \int_0^\infty e^{-t}.\frac{1}{4t^{3/4}}dt$$

x	0	∞
t	0	∞

$$= \frac{1}{4}\int_0^\infty e^{-t}.t^{-3/4}dt$$

$$= \frac{1}{4}\left|\overline{-\frac{3}{4}+1}\right. = \frac{1}{4}\left|\frac{1}{4}\right.$$

$$dx = 3t^2\,dt$$

x	0	∞
t	0	∞

$$I = \int_0^\infty e^{-t} \cdot t^{3/2} \cdot 3t^2\,dt$$

$$= 3\int_0^\infty e^{-t}\, t^{7/2}\,dt$$

$$= 3\,\Gamma\left(\tfrac{7}{2}+1\right) = 3 \cdot \Gamma\left(\tfrac{9}{2}\right)$$

$$= 3 \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{315}{16}\,\Gamma\left(\tfrac{1}{2}\right) = \frac{315}{16}\sqrt{\pi}$$

$$\int_0^\infty x^2 e^{-x^4}\,dx \cdot \int_0^\infty e^{-x^4}\,dx = \frac{\pi}{8\sqrt{2}}$$

Let $x^4 = t \quad \therefore x = t^{1/4} \quad \therefore 4x^3\,dx = dt$

$$\therefore dx = \frac{dt}{4x^3} = \frac{dt}{4t^{3/4}}$$

x	0	∞
t	0	∞

$$\int_0^\infty t^{1/2}\, e^{-t}\left(\frac{dt}{4t^{3/4}}\right) \int_0^\infty e^{-t}\left(\frac{dt}{4t^{3/4}}\right)$$

$$\int_0^\infty e^{-t}\cdot t^{\frac{1}{2}-\frac{3}{4}}\,dt \int_0^\infty e^{-t}\cdot t^{-3/4}\,dt$$

$$\int_0^\infty e^{-t}\cdot t^{-1/4}\,dt \int_0^\infty e^{-t}\, t^{-3/4}\,dt$$

$$\int_0^\infty e^{-t}\, t^{-1/4}\,dt \quad \Gamma\left(-\frac{3}{4}+1\right)$$

$$\Gamma\frac{1}{4}\cdot\Gamma\left(-\frac{1}{4}+1\right) = \frac{1}{16}\,\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$$

$$\Gamma\frac{1}{4}\,\Gamma\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{1}{16}\left(\frac{\sqrt{\pi}}{2^{1/2-1}}\right)\Gamma\frac{1}{2}$$

$\left\{\text{Assume } \Gamma m\cdot\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\cdot\Gamma 2m\right.$

$$= \frac{1}{16}\,\frac{\sqrt{\pi}}{2^{-1/2}}\cdot\sqrt{\pi} = \frac{\sqrt{2}\,\pi}{16} = \frac{\sqrt{2}\,\pi}{8.2}$$

$$= \frac{\pi}{8\sqrt{2}}$$

$\underline{5}$ S.T. $\displaystyle\int_0^\infty \sqrt{y}\cdot e^{-y^2}\,dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy = \frac{\pi}{2\sqrt{2}}$

Let $y^2 = t \quad \therefore y = t^{1/2} \quad \therefore \sqrt{y} = t^{1/4}$

$$\therefore dy = \frac{1}{2}t^{-1/2}\,dt$$

$$I = \int_0^\infty t^{1/4}\cdot e^{-t}\left(\frac{1}{2}t^{-1/2}\,dt\right)\int \frac{e^{-t}}{t^{1/4}}\left(\frac{1}{2}t^{-1/2}\,dt\right)$$

$$= \frac{1}{4}\int_0^\infty e^{-t}\, t^{-1/4}\,dt \int_0^\infty e^{-t}\cdot t^{-3/4}\,dt$$

$$= \frac{1}{4}\,\Gamma\left(-\frac{1}{4}+1\right)\cdot\Gamma\left(-\frac{3}{4}+1\right) = \frac{1}{4}\,\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}$$

$$= \frac{1}{4}\left(\pi\sqrt{2}\right) \quad \left\{\text{from above quest } \Gamma\frac{1}{4}\cdot\Gamma\frac{3}{4} = \pi\sqrt{2}\right)$$

$$\int_0^\infty x^2 \cdot e^{-x^4}\, dx \cdot \int_0^\infty e^{-x^4}\, dx = \frac{\pi}{8\sqrt{2}}$$

Put $x^4 = t \quad \therefore x = t^{1/4} \quad \therefore 4x^3\, dx = dt$

$$\therefore dx = \frac{dt}{4x^3} = \frac{dt}{4t^{3/4}}$$

x	0	∞
t	0	∞

$$\int_0^\infty t^{1/2} \cdot e^{-t} \left(\frac{dt}{4t^{3/4}}\right) \int_0^\infty e^{-t} \left(\frac{dt}{4t^{3/4}}\right)$$

$$\int_0^\infty e^{-t} \cdot t^{\frac{1}{2}-\frac{3}{4}}\, dt \int_0^\infty e^{-t} \cdot t^{-3/4}\, dt$$

$$\int_0^\infty e^{-t} \cdot t^{-1/4}\, dt \int_0^\infty e^{-t}\, t^{-3/4}\, dt$$

$$\int_0^\infty e^{-t}\, t^{-1/4}\, dt \quad \Gamma\left(\frac{-3}{4}+1\right)$$

$$\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{-1}{4}+1\right) = \frac{1}{16}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{1}{16}\left(\frac{\sqrt{\pi}}{2^{1/2-1}}\right)\Gamma\left(\frac{1}{2}\right)$$

$\{$Assume $\Gamma m \cdot \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \Gamma 2m$

$$= \frac{1}{16} \frac{\sqrt{\pi}}{2^{-1/2}} \cdot \sqrt{\pi} = \frac{\sqrt{2}\pi}{16} = \frac{\sqrt{2}\pi}{8 \cdot 2}$$

$$= \frac{\pi}{8\sqrt{2}}$$

$\underline{5}$. S.T. $\int_0^\infty \sqrt{y} \cdot e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} \cdot dy = \frac{\pi}{2\sqrt{2}}$

Let $y^2 = t \quad \therefore y = t^{1/2} \quad \therefore \sqrt{y} = t^{1/4}$

$$\therefore dy = \frac{1}{2} t^{-1/2} dt$$

$$I = \int_0^\infty t^{1/4} \cdot e^{-t} \left(\frac{1}{2} t^{-1/2} dt\right) \int_0^\infty \frac{e^{-t}}{t^{1/4}} \left(\frac{1}{2} t^{-1/2} dt\right)$$

$$= \frac{1}{4} \int_0^\infty e^{-t} t^{-1/4} dt \int_0^\infty e^{-t} \cdot t^{-3/4} dt$$

$$= \frac{1}{4} \Gamma\left(-\frac{1}{4}+1\right) \cdot \Gamma\left(-\frac{3}{4}+1\right) = \frac{1}{4} \Gamma \frac{3}{4} \Gamma \frac{1}{4}$$

$= \frac{1}{4} \left(\pi\sqrt{2}\right)$ $\{$from above quest$\}$

$$\Gamma \frac{1}{4} \cdot \Gamma \frac{3}{4} = \pi\sqrt{2}$$

$$= \int_0^\infty x^2 e^{-h^2x^2} dx$$

Let $h^2x^2 = t \quad \therefore x^2 = t/h^2$

x	0	∞
t	0	∞

$\therefore 2h^2 x\,dx = dt$

$$\therefore dx = \frac{dt}{2h^2 \cdot x} = \frac{dt}{2h^2 \cdot \sqrt{t}}\,h = \frac{dt}{2h\sqrt{t}}$$

$$I = \int_0^\infty \frac{t}{h^2} \cdot e^{-t} \left(\frac{dt}{2h\sqrt{t}}\right)$$

$$= \int_0^\infty \frac{1}{2h^3} \left(e^{-t} \cdot t^{1/2}\,dt\right)$$

$$= \frac{1}{2h^3} \int_0^\infty e^{-t} \cdot t^{1/2}\,dt$$

$$= \frac{1}{2h^3} \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2h^3} \cdot \Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{2h^3} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{4h^3} \cdot \sqrt{\pi} = \frac{\sqrt{\pi}}{4h^3}$$

Type 4: Integrals of type:

$$\int_0^\infty \frac{x^a}{a^x}\,dx$$ can be solved

Substitute $a^x = e^t$

Eg 1. $$\int_0^\infty \frac{x^4}{4^x}\,dx$$

Let $4^x = e^t$

$x\log 4 = t\log e$

$\therefore x = \dfrac{t}{\log 4}$ $\quad\therefore dx = \dfrac{dt}{\log 4}$

$$I = \int_0^\infty e^{-t}\left(\frac{t}{\log 4}\right)^4 \cdot \frac{dt}{\log 4}$$

$$I = \frac{1}{(\log 4)^5}\int_0^\infty e^{-t}\,t^4\,dt$$

$$= \frac{1}{(\log 4)^5}\cdot \Gamma 5 = \frac{4!}{(\log 4)^5} = \frac{24}{(\log 4)^5}$$

S.T. $\int_0^{\infty} \frac{dz}{3^{4z^2}} = \frac{1}{4}\frac{\sqrt{\pi}}{\sqrt{\log 3}}$

Let $3^{4z^2} = e^t$

$\therefore 4z^2 \log 3 = t \quad \therefore z^2 = t/4\log 3$

$\therefore 2z\,dz = \frac{1}{4\log 3} . dt$

z	0	∞
t	0	∞

$dz = \frac{1}{8\log 3} . \frac{1}{\cancel{z}} . dt = \frac{1}{\cancel{4}8\log 3} \frac{\cancel{2}\sqrt{\log 3}}{\sqrt{t}} . dt$

$I = \int_0^{\infty} \frac{1}{e^t} . \left(\frac{\sqrt{\log 3}}{4\log 3 . t^{1/2}} \right) dt$

$= \int_0^{\infty} e^{-t} . t^{-1/2} . \left(\frac{1}{4\sqrt{\log 3}} \right) dt$

$= \frac{1}{4\sqrt{\log 3}} \Gamma\left(-\frac{1}{2}+1\right) = \frac{1}{4\sqrt{\log 3}} \Gamma\frac{1}{2}$

$= \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$

$$\int_0^\infty \frac{x^7}{7^x}\,dx = \frac{7!}{(\log 7)^8}$$

Let $7^x = e^t$

$\therefore x \log 7 = t \quad \therefore x = t/\log 7$

$\therefore dx = \dfrac{dt}{\log 7}$

$$I = \int_0^\infty \frac{t^7}{(\log 7)^7} \cdot e^{-t} \cdot \frac{dt}{\log 7}$$

$$= \frac{1}{(\log 7)^8} \int_0^\infty e^{-t}\, t^7\, dt$$

$$= \frac{\Gamma 8}{(\log 7)^8} = \frac{7!}{(\log 7)^8}$$

III Integrals of type $\int_0^1 x^a (\log x)^b\,dx$ OR $\int_0^1 x^a \left(\log \frac{1}{x}\right)^b dx$

~~Let~~ Substitute $\log x = -t$

or $\log \frac{1}{x} = t$

1. $I = \int_0^1 \frac{dx}{\sqrt{-\log x}}$

Let $\log x = -t \quad \therefore x = e^{-t}$

$\therefore \frac{1}{x}\,dx = -dt$

$dx = -x\,dt$

$= -e^{-t}\,dt$

x	0	1
t	∞	0

$$\therefore I = \int_{\infty}^{0} \frac{1}{\sqrt{t}} \cdot -e^{-t}\,dt$$

$$= \int_{0}^{\infty} e^{-t}\, t^{-1/2}\,dt$$

$$= \Gamma\left(-\tfrac{1}{2}+1\right) = \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

2. $$I = \int_{0}^{1} \frac{x}{\sqrt{\log \frac{1}{x}}}\,dx$$

Let $\log \frac{1}{x} = t \quad \therefore \frac{1}{x} = e^{t} \quad \therefore x = e^{-t}$

$\therefore dx = -e^{-t}\,dt$

x	0	1
t	∞	0

$$I = \int_{\infty}^{0} \frac{e^{-t}}{\sqrt{t}} \left(-e^{-t}\,dt\right)$$

$$= \int_{0}^{\infty} e^{-2t}\, t^{-1/2}\,dt$$

Let $2t = u$

$\therefore 2\,dt = du$

$\therefore dt = du/2$

t	0	∞
u	0	∞

$$I = \int_0^\infty e^{-u} \frac{u^{-1/2}}{2^{-1/2}} \cdot \frac{du}{2}$$

$$= \int_0^\infty \frac{e^{-u}}{\sqrt{2}} \cdot u^{-1/2}\, du$$

$$= \frac{1}{\sqrt{2}} \cdot \Gamma\left(\frac{-1}{2}+1\right) = \frac{1}{\sqrt{2}} \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{2}}$$

$$I = \int_0^1 \frac{dx}{\sqrt{x \log \frac{1}{x}}} = \sqrt{2\pi}$$

Let $\log \frac{1}{x} = t \quad \therefore \frac{1}{x} = e^t \quad \therefore x = e^{-t}$

$dx = -e^{-t} dt$

x	0	1
t	∞	0

$$= \int_\infty^0 \frac{1}{\sqrt{e^{-t} \cdot t}} \left(-e^{-t} dt\right)$$

$$= \int_0^\infty e^{-t} \left(e^{t/2} \cdot t^{1/2}\right) dt$$

$$= \int_0^\infty e^{-t/2} \cdot t^{1/2} dt$$

Let $\frac{t}{2} = u \quad \therefore dt = 2du$

$$= \int_0^\infty e^{-u} (2u)^{1/2} (2du)$$

t	0	∞
u	0	∞

$$I = 2\sqrt{2} \int_0^\infty e^{-u} \cdot u^{1/2} du$$

$$= 2\sqrt{2}\, \Gamma\left(\tfrac{1}{2}+1\right) = 2\sqrt{2}\, \Gamma\left(\tfrac{3}{2}\right)$$

$$= 2\sqrt{2}\, \tfrac{1}{2} \Gamma\left(\tfrac{1}{2}\right) = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}$$

9. $$\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma n.$$

Let $\log \frac{1}{x} = t \quad \therefore \frac{1}{x} = e^t \quad \therefore x = e^{-t}$

$\therefore dx = -e^{-t} dt$

x	0	1
t	∞	0

$$I = \int_\infty^0 t^{n-1} \left(-e^{-t} dt\right)$$

$$= \int_0^\infty e^{-t} \cdot t^{n-1} dt = \Gamma(n-1+1) = \Gamma n$$

$$\text{I.}\quad \int_0^1 \frac{dx}{\sqrt{x\log\frac{1}{x}}} = \sqrt{2\pi}$$

$t \quad \log\frac{1}{x} = t \quad \therefore \frac{1}{x} = e^{t} \quad \therefore x = e^{-t}$

$x = -e^{-t}\,dt$

x	0	1
t	∞	0

$$= \int_\infty^0 \frac{1}{\sqrt{e^{-t}\cdot t}}\left(-e^{-t}dt\right)$$

$$\int_0^\infty e^{-t}\left(e^{t/2}\cdot t^{1/2}\right)dt$$

$$\int_0^\infty e^{-t/2}\cdot t^{1/2}\,dt$$

Let $\frac{t}{2} = u \quad \therefore dt = 2du$

$$= \int_0^\infty e^{-u}(2u)^{1/2}(2du)$$

t	0	∞
u	0	∞

$$I = 2\sqrt{2}\int_0^\infty e^{-u} \cdot u^{1/2}\,du$$

$$= 2\sqrt{2}\,\Gamma\left(\tfrac{1}{2}+1\right) = 2\sqrt{2}\,\Gamma\left(\tfrac{3}{2}\right)$$

$$= 2\sqrt{2}\,\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \sqrt{2}\sqrt{\pi} = \sqrt{2\pi}$$

9 $\displaystyle\int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \Gamma n.$

Let $\log\frac{1}{x} = t \quad \therefore \frac{1}{x} = e^{t} \quad \therefore x = e^{-t}$

x	0	1
t	∞	0

$\therefore dx = -e^{-t}dt$

$$I = \int_\infty^0 t^{n-1}\left(-e^{-t}dt\right)$$

$$= \int_0^\infty e^{-t} \cdot t^{n-1}\,dt = \Gamma(n-1+1) = \Gamma n$$

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BETA FUNCTION:-

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$
$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

Proof:-

I.P.T. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$

Let $I = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$= \int_0^1 (1-x)^{m-1}(1-1+x)^{n-1}\,dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{m-1}\cdot x^{n-1}\,dx$$

$$\therefore B(m,n) = B(n,m)$$

11 P.T. $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Let $x = \sin^2\theta$ $\therefore dx = 2\sin\theta\cos\theta \, d\theta$

x	0	1
θ	0	$\pi/2$

$\therefore \beta(m,n) = 2\int_0^{\pi/2} (\sin^2\theta)^{m-1} \cdot (1-\sin^2\theta)^{n-1} \sin\theta\cos\theta \, d\theta$

$\cancel{= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta}$

$= 2\int_0^{\pi/2} (\sin\theta)^{2m-2} (\cos\theta)^{2n-2} \sin\theta\cos\theta \, d\theta$

$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta$

Let $2m-1 = p$; $2n-1 = q$.

$\therefore m = \frac{p+1}{2}$; $n = \frac{q+1}{2}$

$$\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta$$

$$\therefore \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

III Relation between β & γ

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

{Proof in Double Integration}

IV Prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

If $p=0,\ q=0$

$$\int_0^{\pi/2} 1 . d\theta = \frac{1}{2} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1}$$

$$\frac{\pi}{2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$$

$$\therefore \Gamma\frac{1}{2} = \sqrt{\pi}$$

Type I: Int. of the type

$$\int_0^1 x^m (1-x^n)^p\, dx$$ can be

solved by subs. $x^n = t$

Q1 $I = \int_0^1 x^3(1-\sqrt{x})^5\,dx$

Let $\sqrt{x} = t$

x	0	1
t	0	1

$\therefore x = t^2 \quad \therefore dx = 2t\,dt$

$I = \int_0^1 t^6(1-t)^5\, 2t\,dt$

$= 2\int_0^1 t^7(1-t)^5\,dt$

$= 2\beta(8,6) = 2\frac{\Gamma 8\,\Gamma 6}{\Gamma 14}$

$= 2\cdot\frac{7!\,5!}{13!}$

$= \frac{2\times\cancel{7\times6}\times5\times4\times3\times2}{13\times12\times11\times10\times9\times8\,4} = \frac{1}{5148}$

$\begin{array}{r} 143 \\ \times 36 \\ \hline 858 \\ 429\times \\ \hline 5148 \end{array}$

Q1 $I = \int_0^1 x^3(1-\sqrt{x})^5\,dx$

Let $\sqrt{x} = t$

x	0	1
t	0	1

$\therefore x = t^2 \quad \therefore dx = 2t\,dt$

$I = \int_0^1 t^6(1-t)^5\,2t\,dt$

$= 2\int_0^1 t^7(1-t)^5\,dt$

$= 2\beta(8,6) = 2\dfrac{\Gamma 8\,\Gamma 6}{\Gamma 14}$

$= 2\dfrac{7!\,5!}{13!}$

$= \dfrac{2\times\cancel{7\times 6}\times 5\times 4\times 3\times 2}{13\times 12\times 11\times 10\times 9\times 8\,4} = \dfrac{1}{5148}$

$$\begin{array}{r} 143 \\ \times\ 36 \\ \hline 858 \\ 429\times \\ \hline 5148 \end{array}$$

Q P.T. $\int_0^{\infty} \left(\frac{x}{1+x^2}\right)^6 dx = \frac{3\pi}{512}$

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

x	0	∞
θ	0	$\pi/2$

$I = \int_0^{\pi/2} \frac{\tan^6\theta}{(1+\tan^2\theta)^6} \sec^2\theta\, d\theta$

$= \int_0^{\pi/2} \frac{\tan^6\theta \sec^2\theta}{\sec^{12}\theta}\, d\theta$

$= \int_0^{\pi/2} \frac{\sin^6\theta}{\cos^6\theta} \cos^{10}\theta\, d\theta$

$= \int_0^{\pi/2} \sin^6\theta \cdot \cos^4\theta\, d\theta$

$= \frac{1}{2} B\left(\frac{6+1}{2}, \frac{4+1}{2}\right) = \frac{1}{2} B\left(\frac{7}{2}, \frac{5}{2}\right)$

$$I = \frac{1}{2}\,\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma 6}$$

$$= \frac{1}{2} \times \frac{\frac{\cancel{5}}{2} \times \frac{\cancel{3}}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}}{\cancel{5} \times 4 \times \cancel{3} \times 2}$$

$$= \frac{3\pi}{512}$$

5. S.T. $\displaystyle\int_0^\infty \frac{x^{3/2}}{(1+x)^5}\,dx = \frac{3\pi}{128}$

Let $x = \tan^2\theta$

$\therefore dx = 2\tan\theta\,\sec^2\theta\,d\theta$

x	0	∞
θ	0	$\pi/2$

$$I = \int_0^{\pi/2} \frac{\tan^3\theta}{(\sec^2\theta)^5} \cdot 2\tan\theta\,\sec^2\theta\,d\theta$$

$$I = 2\int_0^{\pi/2} \frac{\tan^4\theta}{\sec^8\theta}\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^4\theta \cdot \cos^4\theta \, d\theta$$

$$= \frac{2}{2}\beta\left(\frac{4+1}{2}, \frac{4+1}{2}\right) = \frac{2}{2}\beta\left(\frac{5}{2}, \frac{5}{2}\right)$$

$$= \frac{2}{2}\,\frac{\Gamma\frac{5}{2}\,\Gamma\frac{5}{2}}{\Gamma 5} = \frac{\cancel{2}}{2}\,\frac{\frac{\cancel{3}}{2}\cdot\frac{1}{\cancel{2}}\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{4\times\cancel{3}\times\cancel{2}}$$

$$= \frac{3\pi}{128}$$

6 S.T. $\displaystyle\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}\, dx = \frac{a\sqrt{\pi}\,\Gamma\frac{5}{6}}{\Gamma\frac{1}{3}}$

$$I = \int_0^a x^{3/2}(a^3-x^3)^{-1/2}\, dx$$

$$= a^{-3/2}\int_0^a x^{3/2}\left(1-\frac{x^3}{a^3}\right)^{-1/2} dx$$

Let $\frac{x^3}{a^3} = t$ $\therefore x = at^{1/3}$

$\therefore dx = \frac{1}{3} a t^{-2/3} dt$

x	0	a
t	0	1

$I = \int_0^1 \cancel{a^{-3/2}} \, \cancel{a^{3/2}} t^{3/3 \cdot 2} (1-t)^{-1/2} \frac{1}{3} a t^{-2/3} dt$

$I = \frac{a}{3} \int_0^1 t^{-1/6} (1-t)^{-1/2} dt$

$= \frac{a}{3} \beta\left(\frac{5}{6}, \frac{1}{2}\right) = \frac{a}{3} \frac{\Gamma\frac{5}{6} \Gamma\frac{1}{2}}{\Gamma\frac{4}{3}}$

$= \frac{a}{\cancel{3}} \frac{\Gamma\frac{5}{6} \sqrt{\pi}}{\frac{1}{\cancel{3}} \Gamma\frac{1}{3}} = \frac{a \sqrt{\pi}\, \Gamma\frac{5}{6}}{\Gamma\frac{1}{3}}$

7 S.T. $\int_0^1 \frac{dx}{\sqrt[n]{1-x^n}} = \Gamma\left(1-\frac{1}{n}\right) \Gamma\left(1+\frac{1}{n}\right)$

Let $x^n = t$ $\therefore x = t^{1/n}$ $\therefore dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

x	0	1
t	0	1

$$I = \int_0^1 (1-x^n)^{-1/n} \cdot dx$$

$$= \int_0^1 (1-t)^{-\frac{1}{n}} \cdot \frac{1}{n} t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{-\frac{1}{n}} dt$$

$$= \frac{1}{n} \beta\left(\frac{1}{n}, 1-\frac{1}{n}\right)$$

$$= \frac{1}{n} \frac{\Gamma\frac{1}{n} \, \Gamma\left(1-\frac{1}{n}\right)}{\Gamma 1} = \Gamma\left(1-\frac{1}{n}\right) \cdot \frac{1}{n} \Gamma\frac{1}{n}$$

$$= \Gamma\left(1-\frac{1}{n}\right) \Gamma\left(\frac{1}{n}+1\right)$$

8 $\int_0^1 x^{m-1}(1-x^2)^{n-1} dx = \frac{1}{2} \beta\left(\frac{m}{2}, n\right)$

Let $x^2 = t$ $\therefore x = t^{1/2}$ $\therefore dx = \frac{1}{2} t^{-1/2} dt$

x	0	1
t	0	1

$$I = \int_0^1 t^{\frac{m-1}{2}} (1-t)^{n-1} \cdot \frac{1}{2} t^{-1/2} \, dt$$

$$= \frac{1}{2} \int_0^1 t^{\frac{m}{2}-\frac{1}{2}-\frac{1}{2}} (1-t)^{n-1} \, dt$$

$$= \frac{1}{2} \int_0^1 t^{\frac{m}{2}-1} (1-t)^{n-1} \, dt$$

$$= \frac{1}{2} \beta\left(\frac{m}{2}, n\right)$$

Q $$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\frac{1}{n}}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

Let $x^n = t$ $\therefore x = t^{1/n}$ $\therefore dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

x	0	1
t	0	1

$$I = \int_0^1 (1-x^n)^{-1/2} \, dx$$

$$= \int_0^1 (1-t)^{-1/2} \cdot \frac{1}{n} t^{\frac{1}{n}-1} \, dt$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{-1/2} \, dt$$

$$= \frac{1}{n} \beta\left(\frac{1}{n}, \frac{1}{2}\right)$$

$$= \frac{1}{n} \frac{\Gamma\frac{1}{n}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{n} \frac{\Gamma\frac{1}{n}}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

10 $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}} = \frac{128}{35}$

Let $x^{1/4} = t \quad \therefore x = t^4 \quad \therefore dx = 4t^3 dt$

x	0	1
t	0	1

$$I = 4\int_0^1 (1-t)^{-1/2} t^3 dt$$

$$= 4\,\beta\left(4, \frac{1}{2}\right) = 4 \frac{\Gamma 4\,\Gamma\frac{1}{2}}{\Gamma\frac{9}{2}}$$

$$= \frac{4 \times \cancel{3} \times \cancel{3} \times 2 \times \sqrt{\pi}}{\frac{7}{2} \times \frac{5}{2} \times \frac{\cancel{3}}{2} \times \frac{1}{2} \times \sqrt{\pi}} = \frac{128}{35}$$

Eg | S.T. $\beta(x+1, y) = \frac{x}{x+y} \beta(x, y)$

$LHS = \beta(x+1, y)$

$= \frac{\Gamma(x+1)\,\Gamma(y)}{\Gamma(x+y+1)}$

$= \frac{x\,\Gamma(x)\cdot\Gamma(y)}{(x+y)\,\Gamma(x+y)} = \frac{x}{x+y} \beta(x, y)$

Eg | S.T. $\beta(x+1, y) = \frac{x}{x+y} \beta(x, y)$

$LHS = \beta(x+1, y)$

$= \frac{\Gamma(x+1)\,\Gamma(y)}{\Gamma(x+y+1)}$

$= \frac{x\,\Gamma(x)\cdot\Gamma(y)}{(x+y)\,\Gamma(x+y)} = \frac{x}{x+y} \beta(x, y)$

4 $\beta(m,n) = \beta(m,n+1) + \beta(m+1,n)$

$$RHS = \frac{\sqrt{m}\,\sqrt{n+1}}{\sqrt{m+n+1}} + \frac{\sqrt{m+1}\,.\,\sqrt{n}}{\sqrt{m+n+1}}$$

$$= \frac{\sqrt{m}\,.\,n\sqrt{n} + m\sqrt{m}\,\sqrt{n}}{\sqrt{m+n+1}}$$

$$= \sqrt{m}\,\sqrt{n}\left(\frac{n+m}{(m+n)\sqrt{m+n}}\right)$$

$$= \frac{\sqrt{m}\,\sqrt{n}}{\sqrt{m+n}} = \beta(m,n)$$

eg 1 $\quad I = \int_3^7 \sqrt[4]{(7-x)(x-3)}\,dx$

Let $x-3 = (7-3)t$

$\therefore x-3 = 4t \quad \therefore x = 3+4t$

$\therefore dx = 4\,dt$

$7-x = 4-4t$

x	3	7
t	0	1

$$I = \int_0^1 \sqrt[4]{(4-4t)(4t)}\,.\,4dt$$

$$= \int_0^1 2(1-t)^{1/4}.\,t^{1/4}.\,4\,dt$$

$$= 8\int_0^1 t^{1/4}(1-t)^{1/4}\,dt$$

$$= 8\,\beta\left(\frac{5}{4},\frac{5}{4}\right) = 8\,\frac{\Gamma\frac{5}{4}\,\Gamma\frac{5}{4}}{\Gamma\frac{5}{2}}$$

$$= 8.\,\frac{\frac{1}{4}\Gamma\frac{1}{4}\,.\,\frac{1}{4}\Gamma\frac{1}{4}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$$

$$= \frac{2}{3}\frac{\left(\Gamma\frac{1}{4}\right)^2}{\Gamma\frac{1}{2}} = \frac{2}{3\sqrt{\pi}}\left(\Gamma\frac{1}{4}\right)^2$$

11 ST $\int_a^b \sqrt[n]{(x-a)(b-x)}\,dx = (b-a)^{2/n+1}\,\beta\left(\frac{1}{n}+1, \frac{1}{n}+1\right)$

Let $x-a=(b-a)t$

$\therefore dx=(b-a)dt$

$b-x=(b-a)(1-t)$

x	a	b
t	0	1

$$I=\int_0^1 \sqrt[n]{(b-a)t\,(b-a)(1-t)}\,(b-a)\,dt$$

$$= (b-a)^{\frac{2}{n}+1}\int_0^1 t^{1/n}(1-t)^{1/n}$$

$$= (b-a)^{2/n+1}\cdot\beta\left(\frac{1}{n}+1, \frac{1}{n}+1\right)$$

Duplication formula for Gamma funct

P.T. $\Gamma m \, \Gamma m+\frac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma 2m$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}$$

Let $p=q$ then

$$\text{et } I = \int_0^{\pi/2} \sin^p\theta \cos^p\theta \, d\theta = \frac{1}{2} \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{p+1}{2}}{\Gamma p+1} \quad \text{— (I)}$$

$$I = \frac{1}{2^p} \int_0^{\pi/2} (2\sin\theta\cos\theta)^p \, d\theta = \frac{1}{2^p} \int_0^{\pi/2} (\sin 2\theta)^p \, d\theta$$

Let $2\theta = t$ $\therefore d\theta = dt/2$

θ	0	$\pi/2$
t	0	π

$$I = \frac{1}{2^p} \int_0^{\pi} (\sin t)^p \frac{dt}{2}$$

$$I = \frac{1}{2.2^p} \int_0^{\pi} (\sin t)^p \, dt$$

$$= \frac{1}{\cancel{2}.2^p} \cdot \cancel{2} \int_0^{\pi/2} (\sin t)^p \, dt \quad \because F$$

$$= \frac{1}{2^p} \int_0^{\pi/2} \sin^p t \cdot \cos^0 t \, dt$$

$$= \frac{1}{2^p} \cdot \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2^p} \cdot \frac{1}{2} \frac{\left\lceil \frac{p+1}{2} \right. \left\lceil \frac{1}{2} \right.}{\left\lceil \frac{p+1}{2} + \frac{1}{2} \right.} \quad \text{--- (II)}$$

From (I) & (II)

$$\cancel{\frac{1}{2}} \frac{\cancel{\left\lceil \frac{p+1}{2} \right.} \left\lceil \frac{p+1}{2} \right.}{\left\lceil p+1 \right.} = \frac{1}{2^p} \cdot \cancel{\frac{1}{2}} \frac{\cancel{\left\lceil \frac{p+1}{2} \right.} \cdot \sqrt{\pi}}{\left\lceil \frac{p+1}{2} + \frac{1}{2} \right.}$$

Let $\frac{p+1}{2} = m \quad \therefore p = 2m - 1$

$$\therefore \frac{\Gamma m}{\Gamma 2m} = \frac{1}{2^p} \frac{\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\therefore \Gamma m \, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} \, \Gamma 2m$$

Eg 1 $I = \int_0^\infty \frac{dx}{x^4+1}$ Let $x^2 = \tan\theta$

$\therefore x = \sqrt{\tan\theta}$

x	0	∞
θ	0	$\pi/2$

$\therefore dx = \frac{1}{2} \cdot \frac{1}{\sqrt{\tan\theta}} \cdot \sec^2\theta \, d\theta$

$$I = \int_0^{\pi/2} \frac{1}{1+\tan^2\theta} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\tan\theta}} \cdot \sec^2\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$$= \frac{1}{2} \cdot \frac{1}{2} \, B\left(\frac{1/2}{2}, \frac{3/2}{2}\right)$$

$$= \frac{1}{4} \, B\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{4} \left(\Gamma\frac{1}{4} \, \Gamma\frac{3}{4}\right) \Big/ \Gamma 1$$

$$\therefore I = \frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)$$

$$= \frac{1}{4}\cdot\frac{\sqrt{\pi}}{2^{1/2-1}}\,\Gamma\left(2\cdot\frac{1}{4}\right)$$

$$= \frac{1}{4}\,\frac{\sqrt{\pi}}{2^{-1/2}}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{4}\cdot\sqrt{\pi}\,\sqrt{\pi}$$

$$= \frac{\pi}{2\sqrt{2}}$$

Eg 2 $\beta(m, m) = \dfrac{\sqrt{\pi}}{2^{2m-1}}\,\dfrac{\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)}$

By Duplication formula,

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\cdot\Gamma(2m)$$

$$\frac{\Gamma(m)}{\Gamma(2m)} = \frac{\sqrt{\pi}}{2^{2m-1}}\cdot\frac{1}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\frac{\Gamma m \, \Gamma m}{\Gamma 2m} = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \frac{\Gamma m}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\therefore \beta(m,m) = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \frac{\Gamma m}{\Gamma\left(m+\frac{1}{2}\right)}$$

Eg 3 P.T. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{\left(\Gamma\frac{1}{4}\right)^2}{4\sqrt{\pi}}$

Let $\displaystyle I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$

Let $\frac{1}{2}\sin^2\theta = \sin^2\phi$

θ	0	$\pi/2$
ϕ	0	$\pi/4$

$\sin\theta = \sqrt{2}\sin\phi$

$\cos\theta \, d\theta = \sqrt{2}\cos\phi \, d\phi$

$$d\theta = \frac{\sqrt{2}\cos\phi \, d\phi}{\sqrt{1-\sin^2\theta}} = \frac{\sqrt{2}\cos\phi \, d\phi}{\sqrt{1-2\sin^2\phi}}$$

$$\frac{\Gamma m\,\Gamma m}{\Gamma 2m} = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \frac{\Gamma m}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\therefore \beta(m,m) = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \frac{\Gamma m}{\Gamma\left(m+\frac{1}{2}\right)}$$

Eg 3 P.T. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{\left(\Gamma\frac{1}{4}\right)^2}{4\sqrt{\pi}}$

Let $\displaystyle I = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$

Let $\frac{1}{2}\sin^2\theta = \sin^2\phi$

θ	0	$\pi/2$
ϕ	0	$\pi/4$

$\sin\theta = \sqrt{2}\sin\phi$

$\cos\theta\, d\theta = \sqrt{2}\cos\phi\, d\phi$

$$d\theta = \frac{\sqrt{2}\cos\phi\, d\phi}{\sqrt{1-\sin^2\theta}} = \frac{\sqrt{2}\cos\phi\, d\phi}{\sqrt{1-2\sin^2\phi}}$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}} \cdot \frac{\Gamma\frac{1}{4}}{\Gamma\frac{1}{4}}$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{\pi}\left(\Gamma\frac{1}{4}\right)^2}{\Gamma\frac{1}{4}\,\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{\left(\Gamma\frac{1}{4}\right)^2 2^{2\times\frac{1}{4}-1}}{\sqrt{\pi}\,\Gamma\cancel{2}\cdot\frac{1}{4}} = \frac{\sqrt{\pi}}{2\sqrt{2}} \frac{\left(\Gamma\frac{1}{4}\right)^2 2^{-1/2}}{\sqrt{\pi}\,\sqrt{\pi}}$$

$$= \frac{\left(\Gamma\frac{1}{4}\right)^2}{4\sqrt{\pi}}$$

Prob on β & Γ funct

S.T. $\boxed{\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx}$

Let $x = \tan^2\theta$

$\therefore dx = 2\tan\theta\sec^2\theta\,d\theta$

x	0	∞
θ	0	$\pi/2$

$$I = \int_0^{\pi/2} \frac{\tan^{2m-2}\theta}{(1+\tan^2\theta)^{m+n}} \cdot 2\tan\theta \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2m-1}\theta}{\sec^{2m+2n}\theta} \cdot \sec^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \tan^{2m-1}\theta \cdot \sec^{2-2m-2n}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2m-1}\theta}{\cos^{2m-1}\theta} \cdot \frac{1}{\cos^{2-2m-2n}\theta}\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2m-1}\theta}{\cos^{1-2n}\theta} = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= 2 \cdot \frac{1}{2}\beta\left(\frac{2m\cancel{-1}\cancel{+1}}{2}, \frac{2n\cancel{-1}\cancel{+1}}{2}\right)$$

$$= \beta(m,n)$$

2 S.T.

$$\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$$

We have,

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \quad \{\text{from prob. 1}\}$$

$$= \int_0^1 \frac{x^{m-1}\, dx}{(1+x)^{m+n}} + \int_1^\infty \frac{x^{m-1}\, dx}{(1+x)^{m+n}}$$

Let $I = \int_1^{\infty} \frac{x^{m-1}\,dx}{(1+x)^{m+n}}$; Let $x = \frac{1}{t}$

$\therefore dx = -\frac{1}{t^2}\,dt$

x	1	∞
t	1	0

$$I = \int_1^0 \left(\frac{1}{t}\right)^{m-1} \cdot \frac{1}{(1+1/t)^{m+n}} \cdot -\frac{1}{t^2}\,dt$$

$$= \int_0^1 \frac{t^{1-m} \cdot t^{m+n} \cdot t^{-2}}{(1+t)^{m+n}}\,dt = \int_0^1 \frac{t^{n-1}\,dt}{(1+t)^{m+n}}.$$

$$\therefore I = \int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{m+n}} \quad \left\{ \int_a^b f(x)\,dx = \int_a^b f(t)\,dt \right\}$$

$$B(m,n) = \int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{m+n}} + \int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{m+n}}$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$$

3) P.T. $\int_0^\infty \frac{x^8(1-x^6)\,dx}{(1+x)^{24}} = 0$

$$I = \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \beta(9,15) - \beta(9,15)$$

$$= 0$$

$\left\{ \because \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n) \right.$

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Q4. P.T. $\int_0^\infty \frac{(x^6 - x^3)}{(1+x^3)^5} x^2 dx = 0$

Let $x^3 = t \quad \therefore 3x^2 dx = dt \quad \begin{array}{c|c|c} x & 0 & \infty \\ t & 0 & \infty \end{array}$

$\therefore I = \frac{1}{3}\int_0^\infty \frac{(t^2 - t)}{(1+t)^5} dt$

$= \frac{1}{3}\left[\int_0^\infty \frac{t^2 dt}{(1+t)^5} - \int_0^\infty \frac{t}{(1+t)^5} dt\right]$

$= \frac{1}{3}\left[\int_0^\infty \frac{t^{3-1} dt}{(1+t)^{3+2}} - \int_0^\infty \frac{t^{2-1}}{(1+t)^{2+3}} dt\right]$

$= \frac{1}{3}\left[\beta(3,2) - \beta(2,3)\right]$ ← **

$= 0$

5. P.T. $\int_0^1 \frac{x^{m-1} \cdot (1-x)^{n-1}}{(b+cx)^{m+n}} dx = \frac{1}{b^n (b+c)^m} \beta(m,n)$

$I = \frac{1}{c^{m+n}} \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(\frac{b}{c}+x)^{m+n}} dx$ Let $\frac{b}{c} = a$

$$\therefore I = \frac{1}{c^{m+n}} \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx$$

Let $\frac{x}{a+x} = \frac{t}{a+1}$ [To be rem.]

$$\therefore \frac{a+x}{x} = \frac{a+1}{t} \quad \therefore \frac{a}{x}+1 = \frac{a+1}{t}$$

$$\therefore \frac{a}{x} = \frac{a+1}{t} - 1 = \frac{a+1-t}{t}$$

x	0	1
t	0	1

$$\therefore x = \frac{at}{a+1-t} \quad \therefore dx = \frac{(a+1-t)(a) - at(-1)}{(a+1-t)^2} \qquad \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(a+1-t)a + at}{(a+1-t)^2} = \frac{a[a+1-\cancel{t}+\cancel{t}]}{(a+1-t)^2}\,dt$$

$$dx = \frac{a(a+1)}{(a+1-t)^2}\,dt$$

$$1-x = 1 - \frac{at}{a+1-t} = \frac{a+1-t-at}{a+1-t}$$

$$= \frac{(a+1)-t(a+1)}{a+1-t} = \frac{(a+1)(1-t)}{a+1-t}$$

$$\therefore\ a + x = a + \frac{at}{a+1-t} = \frac{a^2 + a - \cancel{at} + \cancel{at}}{(a+1-t)} = \frac{a(a+1)}{a+1-t}$$

$$\cancel{\therefore I = \frac{1}{c^{m+n}} \int_0^1 \left(\frac{at}{a+1-t}\right)^{m-1} \left(1 - \frac{at}{a+1-t}\right)^{n-1}}$$

$$I = \frac{1}{c^{m+n}} \int_0^1 \left(\frac{at}{a+1-t}\right)^{m-1} \left(\frac{(a+1)(1-t)}{a+1-t}\right)^{n-1} \cdot \left(\frac{a+1-t}{a(a+1)}\right)^{m+n} \cdot \frac{a(a+1)\,dt}{(a+1-t)^2}$$

$$= \frac{1}{c^{m+n}} \int_0^1 \frac{a^{m-1} \cdot t^{m-1}}{(a+1-t)^{m-1}} \cdot \frac{(a+1)^{n-1}(1-t)^{n-1}}{(a+1-t)^{n-1}} \cdot \frac{(a+1-t)^{m+n}}{a^{m+n}(a+1)^{m+n}} \cdot \frac{a\,(a+1)\,dt}{(a+1-t)^2}$$

$$= \frac{1}{c^{m+n}} \int_0^1 a^{m-1+1-m-n} \cdot (a+1)^{n-1+1-m-n} \cdot t^{m-1}(1-t)^{n-1} \cdot (a+1-t)^{m+n-m+1-n+1-2}\,dt$$

$$= \frac{a^{-n} \cdot (a+1)^{-m}}{c^{m+n}} \int_0^1 t^{m-1}(1-t)^{n-1}\,dt$$

$$= \frac{1}{c^{m+n}(a^n)(a+1)^m}\, \beta(m,n)$$

$$= \frac{1}{c^m \cdot c^n \left(\frac{b^n}{c^n}\right)(b+c)^m \cdot \frac{1}{c^m}} \cdot \beta(m,n) = \frac{1}{b^n(b+c)^m}\, \beta(m,n)$$

6. S.T. $\int_1^\infty \frac{dx}{x^{p+1}(x-1)^q} = \beta(p+q, 1-q)$

Let $I = \int_1^\infty \frac{dx}{x^{p+1}(x-1)^q}$

x	1	∞
t	1	0

Let $x = 1/t$ $\therefore dx = -1/t^2 \, dt$

$$I = \int_1^0 \frac{1}{\frac{1}{t^{p+1}}\left(\frac{1-t}{t}\right)^q} \cdot -\frac{1}{t^2} dt$$

$$= \int_0^1 t^{p+1-2+q} (1-t)^{-q} dt$$

$$= \int_0^1 t^{p+q-1} (1-t)^{-q} dt$$

$$= \beta(p+q, -q+1)$$

Eg 7 S.T. $\int_0^{\pi/2} \frac{\sin^{2m-1}x \cos^{2n-1}x}{(a\sin^2 x + b\cos^2 x)^{m+n}}\,dx = \frac{1}{2a^m b^n}\beta(m,n)$

Let

$$I = \int_0^{\pi/2} \frac{\sin^{2m-1}x \cos^{2n-1}x}{(a\sin^2 x + b\cos^2 x)^{m+n}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin^{2m}x \cos^{2n}x \cdot \sin^{-1}x \cos^{-1}x}{(b\cos^2 x)^{m+n}\left[\frac{a}{b}\tan^2 x + 1\right]^{m+n}}\,dx$$

$$= \frac{1}{b^{m+n}} \int_0^{\pi/2} \frac{\cos x}{\sin x \cos^2 x} \frac{\sin^{2m}x}{\cos^{2m}x} \frac{dx}{\left(1+\frac{a}{b}\tan^2 x\right)^{m+n}}$$

$$= \frac{1}{b^{m+n}} \int_0^{\pi/2} \frac{\tan^{2m-1}x}{(1+a/b\tan^2 x)^{m+n}} \sec^2 x\,dx \;\frac{\tan x}{\cancel{\tan x}}$$

Let $a/b\tan^2 x = t$

x	0	$\pi/2$
t	0	∞

$\therefore \tan^2 x = \frac{bt}{a}$ $\therefore 2\tan x\sec^2 x\,dx = \frac{b\,dt}{a}$

$$I = \frac{1}{2b^{m+n}} \int_0^\infty \left(\frac{bt}{a}\right)^{m-1} \frac{dt}{(1+t)^{m+n}} \cdot \frac{b}{a}$$

$$= \frac{b^{m-1}}{2b^{m+n} \cdot a^{m-1}} \frac{b}{a} \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} \, dt$$

$$= \frac{1}{2a^m b^n} \cdot \beta(m, n)$$

8) ST $\displaystyle\int_0^\infty \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4} \beta\left(\frac{n}{2}, \frac{n}{2}\right)$

& find $\displaystyle\int_0^\infty \operatorname{sech}^8 x \, dx$

Let $I = \displaystyle\int_0^\infty \frac{e^{nx} \, dx}{(e^{2x}+1)^n}$

x	0	∞
t	1	∞

Let $e^{2x} = t \quad \therefore e^x = t^{1/2}, e^{nx} = t^{n/2}$

$2x = \log t$

$dx = \dfrac{1}{2t} \, dt.$

$$I = \int_1^\infty \frac{t^{n/2}}{(t+1)^n} \cdot \frac{1}{2t}\, dt$$

t	1	∞
u	1	0

$$= \frac{1}{2}\int_1^\infty \frac{t^{n/2-1}}{(t+1)^n}\, dt \quad \text{—} \quad (I)$$

Let $t = 1/u$

$$\therefore dt = -1/u^2\, du$$

$$= \frac{1}{2}\int_1^0 \frac{(1/u)^{\frac{n}{2}-1}}{\left(1+\frac{1}{u}\right)^n} \cdot -\frac{1}{u^2}\, du$$

$$= \frac{1}{2}\int_0^1 \frac{u^{1-\frac{n}{2}+n-2}}{(1+u)^n}\, du = \frac{1}{2}\int_0^1 \frac{u^{n/2-1}}{(1+u)^n}\, du$$

$$= \frac{1}{2}\int_0^1 \frac{t^{n/2-1}}{(1+t)^n}\, dt \quad \text{—} \quad (II) \quad \left\{\because \int_a^b f(u)\,du = \int_a^b f(t)\,dt\right\}$$

(I) + (II)

$$2I = \frac{1}{2}\left[\int_0^1 \frac{t^{n/2-1}}{(1+t)^n} + \int_1^\infty \frac{t^{n/2-1}}{(1+t)^n}\, dt\right]$$

$$\therefore I = \frac{1}{4}\int_0^\infty \frac{t^{n/2-1}}{(1+t)^n}\, dt = \frac{1}{4}\int_0^\infty \frac{t^{n/2-1}}{(1+t)^{n/2+n/2}}\, dt$$

$$I = \frac{1}{4} \beta\left(\frac{n}{2}, \frac{n}{2}\right)$$

$$I_1 = \int_0^\infty \operatorname{sech}^8 x \, dx = \int_0^\infty \frac{2^8}{(e^x + e^{-x})^8} dx$$

$\therefore n = 8$

$$\therefore I_1 = 2^8 \cdot \frac{1}{4} \beta\left(\frac{8}{2}, \frac{8}{2}\right)$$

$$= 2^8 \cdot \frac{1}{4} \beta(4, 4)$$

$$= 2^8 \cdot \frac{1}{4} \frac{\Gamma 4 \, \Gamma 4}{\Gamma 8} = \frac{16}{35}$$

Ex 6.9

Q1 ST $\int_0^\infty \frac{x^{m-1}\,dx}{(a+bx)^{m+n}} = \frac{1}{a^n b^m}\beta(m,n)$

Let $I = \int_0^\infty \frac{x^{m-1}\,dx}{(a+bx)^{m+n}}$

$= \frac{1}{a^{m+n}}\int_0^\infty \frac{x^{m-1}\,dx}{(1+bx/a)^{m+n}}$ Let $\frac{bx}{a} = t \therefore x = \frac{at}{b}$

$\therefore dx = \frac{a}{b}\,dt$

x	0	∞
t	0	∞

$I = \frac{1}{a^{m+n}}\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} \cdot \frac{a^{m-1}}{b^{m-1}} \cdot \frac{a}{b}\,dt$

$= \frac{a^m}{b^m a^{m+n}}\beta(m,n) = \frac{1}{a^n b^m}\beta(m,n)$