(v)
$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & -1 & 4 \\ 4 & 3 & 2 & 5 \\ 1 & -1 & 4 & -3 \end{bmatrix}$$

$$|A| = 2(-52-4+56) - 1(-26-17+56) + 3(-4+34-28) - 1(14-28+7)$$
$$= 0 - 13 + 6 + 7$$
$$= 0$$

$\because |A| = 0 \therefore A^{-1}$ does not exist

TYPE II: RANK OF A MATRIX & NORMAL FORM:-

Normal form of a matrix is given by

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} I_r & 0 \end{bmatrix}, \begin{bmatrix} I_r \end{bmatrix}$$

row of zeroes

column of zeroes

where I_r is a unit matrix of order r

In the end chp.

Rank of a matrix is the no. of non-zero rows in the canonical (echelon form) or the normal form (order of I_r)
i.e. $\rho(A) = r$

NOTE: If $\rho(A) = 0$, then A is a null matrix

$\rho(A) \geq 1$, then A is not a zero matrix

$\rho(A) = n$, then A is a non-singular matrix of order $n \times n$

$\rho(A) \leq$ min. of m & n for matrix A of order $m \times n$

To convert to normal form we can use row & column transformations.

3 i)

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$$

$R_2 - 3R_1$; $R_3 + R_1$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

$R_2 \times (-1/2)$

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -3/2 & 5 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

$R_1 - 2R_2$; $R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 2 & -7 \\ 0 & 1 & -3/2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

*- Next trans. $C_3 - 2C_1$, $^{*}C_3 + {}^{2}/_{3}C_2$, $C_4 + 7C_1$, $^{*}C_4 \; {}^{-5/2}$

$$A \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2$

(ii)

$$A = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$$

$-(R_1 - R_2)$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$$

$R_2 - 4R_1$; $R_3 - 5R_1$; $R_4 - 10R_1$; $R_5 - 15R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$R_1 - R_2,\ R_3 - R_2,\ R_4 - R_2,\ R_5 - R_2$

$$A \sim \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 + C_1;\ \cancel{C_3 - 2C_2},\ C_4 + 2C_1;\ C_5 + 3C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 - 2C_2,\ C_4 - 3C_2,\ C_5 - 4C_2$

$$A \sim \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

(iii)

$$A = \begin{bmatrix} 2 & -2 & -2 & 0 & 2 \\ 3 & -1 & -1 & 4 & 1 \\ 0 & -1 & 0 & -1 & 2 \\ 1 & 1 & 3 & 2 & -2 \\ 2 & 2 & 4 & 6 & 3 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 2 & -2 \\ 3 & -1 & -1 & 4 & 1 \\ 0 & -1 & 0 & -1 & 2 \\ 2 & -2 & -2 & 0 & 2 \\ 2 & 2 & 4 & 6 & 3 \end{bmatrix}$$

$R_2 - 3R_1 ;\ R_4 - 2R_1 ,\ R_5 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 2 & -2 \\ 0 & -4 & -10 & -2 & 7 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & -4 & -8 & -4 & 6 \\ 0 & 0 & -2 & 2 & 7 \end{bmatrix}$$

$R_2 - 5R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 2 & -2 \\ 0 & 1 & -10 & 3 & -3 \\ 0 & -1 & 0 & -1 & 2 \\ 0 & -4 & -8 & -4 & 6 \\ 0 & 0 & -2 & 2 & 7 \end{bmatrix}$$

$R_1 - R_2$; $R_3 + R_2$; $R_4 + 4R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 13 & -1 & 1 \\ 0 & 1 & -10 & 3 & -3 \\ 0 & 0 & -10 & 2 & -1 \\ 0 & 0 & -48 & 8 & -6 \\ 0 & 0 & -2 & 2 & 7 \end{bmatrix}$$

$R_3 \times (-1/10)$

$$A \sim \begin{bmatrix} 1 & 0 & 13 & -1 & 1 \\ 0 & 1 & -10 & 3 & -3 \\ 0 & 0 & 1 & -1/5 & 1/10 \\ 0 & 0 & -48 & 8 & -6 \\ 0 & 0 & -2 & 2 & 7 \end{bmatrix}$$

$R_1 - 13R_3$; $R_2 + 10R_3$; $R_4 + 48R_3$; $R_5 + 2R_3$

$$\text{Aн}\begin{bmatrix} 1 & 0 & 0 & 8/5 & -3/10 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1/5 & 1/10 \\ 0 & 0 & 0 & -8/5 & -6/5 \\ 0 & 0 & 0 & 8/5 & 36/5 \end{bmatrix}$$

$R_4 \times (-5/8)$

$$\text{Aн}\begin{bmatrix} 1 & 0 & 0 & 8/5 & -3/10 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1/5 & 1/10 \\ 0 & 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 8/5 & 36/5 \end{bmatrix}$$

$R_1 - 8/5 R_4$; $R_2 - R_4$, $R_3 + 1/5 R_4$; $R_5 - 8/5 R_4$

$$\text{Aн}\begin{bmatrix} 1 & 0 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & 0 & -9/4 \\ 0 & 0 & 1 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$R_5 (1/6)$

~~***~~ :- Non-singular matrices P & Q are to be found either: we apply Xform. so that P & Q are upper or lower Δ^{le} matr.

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & 0 & -9/4 \\ 0 & 0 & 1 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 + 3/2 R_5$; $R_2 + 9/4 R_5$; $R_3 - 1/4 R_5$; $R_4 - 3/4 R_5$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = [I_5]$$

$\therefore \rho(A) = 5$

TYPE III :- PAQ form ~~***~~

4P)

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$$

$$A = I_3 A I_4$$

no. of row   no. of columns

$$\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 5 & 1 & 4 & -2 \\ 3 & 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 5R_1$; $R_3 - 3R_1$

$$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 0 & 21 & -51 & 93 \\ 0 & 14 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -5 \\ 1 & 0 & -3 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \times (1/21)$

$$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 0 & 1 & -51/21 & 93/21 \\ 0 & 14 & -34 & 62 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/21 & -\frac{5}{21} \\ 1 & 0 & -3 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

column

$$\begin{bmatrix} 1 & -4 & 11 & -19 \\ 0 & 1 & -51/21 & 93/21 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/21 & -5/21 \\ 1 & -\frac{2}{3} & 1/3 \end{bmatrix} \cdot A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 + 4C_1$

$$\begin{bmatrix} 1 & 0 & 11 & -19 \\ 0 & 1 & -51/21 & 93/21 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/21 & -5/21 \\ 1 & -2/3 & 1/3 \end{bmatrix} A \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 - 11C_1 ; C_4 + 19C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -51/21 & 93/21 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/21 & -5/21 \\ 1 & -2/3 & 1/3 \end{bmatrix} . A \begin{bmatrix} 1 & 4 & -11 & 19 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 + \frac{51}{21} C_2 ; C_4 - \frac{93}{21} C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \overset{P}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/21 & -5/21 \\ 1 & -2/3 & 1/3 \end{bmatrix}} . A \overset{Q}{\begin{bmatrix} 1 & 4 & -\frac{9}{7} & \frac{9}{7} \\ 0 & 1 & \frac{51}{21} & -\frac{93}{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$|P| = -1/21$; $|Q| = $ ——

$\because |P| \neq 0$ & $|Q| \neq 0$ $\therefore$ P & Q are non-singular.

$\rho(A) = \rho(PAQ) = 2$

ii) $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$A = I_3 \cdot A \cdot I_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_1$; $R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - C_1$; $C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \cdot A \cdot \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2(-1/2)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ +1/2 & -1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

~~$C_3 - C_2$~~

$$\cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 1/2 \\ -2 & -1 & 2 \end{bmatrix}}_{P} A \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{Q}}$$

$C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix}}_{P} A \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{Q}$$

$$\rho(A) = \rho(PAQ) = 2$$

TYPE IV : ORTHOGONAL MATRIX

Matrix A is orthogonal if

$$AA' = A'A = I$$

For orthogonal matrix A

$$A^{-1} = A'$$

i)

$$AA' = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A'A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore AA' = A'A = I$

Matrix A is orthogonal

$$A^{-1} = A' = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

ii) $$AA' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & \cos^2\theta + \sin^2\theta & 0 \\ 0 & 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A'A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore AA' = A'A = I$

$\therefore$ Matrix A is orthogonal

$$A^{-1} = A' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

6 iii)

$$AA' = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

$$= \begin{bmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_1l_2+m_1m_2+n_1n_2 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3 \\ l_1l_3+m_1m_3+n_1n_3 & l_2l_3+m_2m_3+n_2n_3 & l_3^2+m_3^2+n_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \{\because A \text{ is ortho}\}$$

$l_1^2+m_1^2+n_1^2 = 1$; $l_1l_2+m_1m_2+n_1n_2 = 0$

$l_2^2+m_2^2+n_2^2 = 1$; $l_1l_3+m_1m_3+n_1n_3 = 0$

$l_3^2+m_3^2+n_3^2 = 1$; $l_2l_3+m_2m_3+n_2n_3 = 0$

$$A'A = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

$$\begin{bmatrix} l_1^2+l_2^2+l_3^2 & l_1m_1+l_2m_2+l_3m_3 & l_1n_1+l_2n_2+l_3n_3 \\ l_1m_1+l_2m_2+l_3m_3 & m_1^2+m_2^2+m_3^2 & m_1n_1+m_2n_2+m_3n_3 \\ l_1n_1+l_2n_2+l_3n_3 & m_1n_1+m_2n_2+m_3n_3 & n_1^2+n_2^2+n_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$l_1^2+l_2^2+l_3^2=1$; $l_1m_1+l_2m_2+l_3m_3=0$

$m_1^2+m_2^2+m_3^2=1$; $l_1n_1+l_2n_2+l_3n_3=0$

$n_1^2+n_2^2+n_3^2=1$; $m_1n_1+m_2n_2+m_3n_3=0$

)

$$AA' = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 4\beta^2+r^2 & 2\beta^2-r^2 & -2\beta^2+r^2 \\ 2\beta^2-r^2 & \alpha^2+\beta^2+r^2 & \alpha^2-\beta^2-r^2 \\ -2\beta^2+r^2 & \alpha^2-\beta^2-r^2 & \alpha^2+\beta^2+r^2 \end{bmatrix}$$

$\therefore AA' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$4\beta^2 + r^2 = 1$

$2\beta^2 - r^2 = 0 \quad \therefore r^2 = 2\beta^2$

$\therefore \beta = \pm \frac{1}{\sqrt{6}} \; ; \; r = \pm \frac{1}{\sqrt{3}}$

$\alpha^2 - \beta^2 - r^2 = 0$

$\therefore \alpha^2 = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \quad \therefore \alpha = \pm \frac{1}{\sqrt{2}}$

LINEAR ALGEBRAIC EQN.

TYPE I : ECHELON OR CANONICAL FORM -

A matrix is converted to ech. or can. form by applying row Xform. and converted the matrix to upper Δ^{le} form, the no. of non-zero rows is the rank of the mat.

1)
$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 - 2R_1;\ R_3 - 3R_1,\ R_4 - 6R_1$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & +3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 - R_3$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

Not req. → ~~$R_1 + R_2$~~ ; $R_3 - 4R_2$; $R_4 - 9R_2$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 - 2R_3$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$ {No. of non-zero rows}

TYPE II: CONSISTENCY & SOLN. OF NON-HOMOGENEOUS EQS.

Step 1: We write the eqs. in matrix form i.e. $AX = D$

Step 2: Write the augmented matrix $[A:D]$

Step 3: Convert the aug. matrix to echelon or canonical form using row Xform.

Step 4: Obtain $\rho(A)$ & $\rho(A:D)$

Step 5: (i) If $\rho(A) \neq \rho(A:D)$ then eqs. are non-consist. & have no soln.

~~Step 6~~ (ii) If $\rho(A) = \rho(A:D) =$ No. of unknowns of the eq. then the eqs. are consistent & have unique soln.

(iii) If $\rho(A) = \rho(A:D) <$ No. of unknowns of the eq. then the eqs. are consistent & have infinitely many soln.

No. of parameter = ~~$\rho(A)$~~ No. of unknown of eq $-$ $\rho(A)$

2i) In matrix form

$$\begin{bmatrix} 2 & 3 & -4 \\ 1 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -5 \end{bmatrix}$$

i.e. $A \cdot X = D$

Augmented matrix is

$$[A : D] = \left[\begin{array}{ccc:c} 2 & 3 & -4 & -2 \\ 1 & -1 & 3 & 4 \\ 3 & 2 & -1 & -5 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 3 & 4 \\ 2 & 3 & -4 & -2 \\ 3 & 2 & -1 & -5 \end{array}\right]$$

$R_2 - 2R_1$; $R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & -1 & 3 & : & 4 \\ 0 & 5 & -10 & : & -10 \\ 0 & 5 & -10 & : & -17 \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 3 & : & 4 \\ 0 & 5 & -10 & : & -10 \\ 0 & 0 & 0 & : & -7 \end{bmatrix}$$

$\rho(A) = 2$; $\rho(A:D) = 3$

$\rho(A) \neq \rho(A:D)$

$\therefore$ Eqns. are not consistent & have no solutions.

ii) In matrix form

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

~~$R_1 \leftrightarrow R_3$~~ i.e $A \cdot X = D$

Augmented matrix is

$$[A:D] = \left[\begin{array}{ccc:c} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{array}\right]$$

$R_2 - 2R_1 \,;\, R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{array}\right]$$

$R_3 - \frac{5}{7} R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & +\frac{8}{7} & -\frac{8}{7} \end{array}\right]$$

$\rho(A) = 3 \,;\, \rho(A:D) = 3$

$\therefore \rho(A) = \rho(A:D) =$ No. of unknown ∴ Eqns. are consistent & have unique so

In matrix form,

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 8/7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -8/7 \end{bmatrix}$$

$\therefore \frac{8}{7}z = -\frac{8}{7}$ $\therefore z = -1$

$-7y - 3z = -11$

$\therefore -7y + 3 = -11$ $\therefore y = 2$

& $x + 2y + z = 4$

$\therefore x + 4 - 1 = 4$ $\therefore x = 1$

ii) In matrix form

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$A \quad X = D$$

Augmented matrix is

$$[A : D] = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$R_1 - R_3$ OR $\frac{R_1}{5}$

$$\sim \left[\begin{array}{ccc|c} -2 & 1 & -3 & -1 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$R_1 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 27 & -1 & 8 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$R_2 - 3R_1 ; R_3 - 7R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 27 & -1 & 8 \\ 0 & -55 & 5 & -15 \\ 0 & -187 & 17 & -51 \end{array}\right]$$

$R_2 \times (-1/55)$

$$\sim\left[\begin{array}{ccc:c} 1 & 27 & -1 & 8 \\ 0 & 1 & -1/11 & 3/11 \\ 0 & -187 & 17 & -51 \end{array}\right]$$

$R_3 + 187R_2$

$$\sim\left[\begin{array}{ccc:c} 1 & 27 & -1 & 8 \\ 0 & 1 & -1/11 & 3/11 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 \; ; \; \rho(A:D) = 2$

$\rho(A) = \rho(A:D) <$ No. of unknowns (3)

$\therefore$ Eqs. are consist. & have infinitely many soln.

No. of param. = No. of unknowns $- \rho(A) = 3-2 = 1$

Let $z = k$; In matrix form

$$\begin{bmatrix} 1 & 27 & -1 \\ 0 & 1 & -1/11 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3/11 \\ 0 \end{bmatrix}$$

$y - \frac{z}{11} = \frac{3}{11} \quad \therefore y = \frac{3}{11} + \frac{k}{11}$

$$x + 27y - z = 8$$

$$x + 27\left(\frac{3+k}{11}\right) - k = 8$$

$$x = k + 8 - \left(\frac{81 + 27k}{11}\right)$$

$$x = \frac{-16k + 7}{11}$$

(v) In matrix form

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

$$A \quad X = D$$

Augmented matrix is

$$[A:D] = \left[\begin{array}{cccc:c} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

$R_1 \times (1/2)$

$$\sim \left[\begin{array}{cccc:c} 1 & 1/2 & 1 & 1/2 & 3 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

$R_2 - 6R_1$; $R_3 - 4R_1$; $R_4 - 2R_1$

$$\sim \left[\begin{array}{cccc:c} 1 & 1/2 & 1 & 1/2 & 3 \\ 0 & -9 & 0 & 9 & 18 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right]$$

$R_2 \times (-1/9)$

$$\sim \left[\begin{array}{cccc:c} 1 & 1/2 & 1 & 1/2 & 3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -5 & -13 \\ 0 & 1 & -3 & 0 & 4 \end{array}\right]$$

$R_3 - R_2$; $R_4 - R_2$

$$\sim \left[\begin{array}{cccc:c} 1 & 1/2 & 1 & 1/2 & 3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & -2 & +1 & 6 \end{array}\right]$$

$R_4 - 3R_3$

$$\sim \left[\begin{array}{cccc:c} 1 & 1/2 & 1 & 1/2 & 3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -4 & -11 \\ 0 & 0 & 0 & 13 & 39 \end{array}\right]$$

$\rho(A) = 4 \; ; \; \rho(A:D) = 4$

$\rho(A) = \rho(A:D) = $ No. of unknown 4

$\therefore$ eqns. are consist. & have unique soln.

In matrix form

$$\begin{bmatrix} 1 & 1/2 & 1 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -11 \\ 39 \end{bmatrix}$$

$13x_4 = 39 \quad \therefore x_4 = 3$

$-x_3 - 4x_4 = -11 \quad \therefore x_3 = -1$

$x_2 - x_4 = -2 \quad \therefore x_2 = 1$

$x_1 + \frac{x_2}{2} + x_3 + \frac{x_4}{2} = 3 \quad \therefore x_1 = 2$

(v) In matrix form

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

$$A \cdot X = D$$

Aug. matrix is

$$[A:D] = \left[\begin{array}{ccc:c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

$R_2 - 3R_1 \; ; \; R_4 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

$(3R_2 + R_3)$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

$R_3 - 10R_2 \,;\, R_4 + 7R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & -87 & 348 \\ 0 & 0 & 62 & -248 \end{array}\right]$$

$R_3 \times (-1/87)$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 1 & -348/87 \\ 0 & 0 & 62 & -248 \end{array}\right]$$

$R_4 - 62R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 3 \; ; \; \rho(A:D) = 3$

$\rho(A) = \rho(A:D) =$ No. of unknowns (3)

$\therefore$ Eqns. are consist. & have unique eqns.

In matrix form -

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -35 \\ -4 \\ 0 \end{bmatrix}$$

$z = -4$

$y + 9z = -35 \; ; \; y = 1$

$x + 2y = 4 \; ; \; x = 2$

(vi) In matrix form

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 2 & -1 & 3 & 0 & 4 \\ 3 & -2 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$A \cdot X = D$$

Augmented matrix is

$$[A \vdots D] = \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 2 & -1 & 3 & 0 & 4 & 2 \\ 3 & -2 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

$R_2 - 2R_1 ; R_3 - 3R_1 ; R_4 - R_1$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 1 & -1 & 4 & -2 & -2 \\ 0 & 1 & 0 & 3 & 0 & -1 \end{array}\right]$$

$R_3 - R_2 ; R_4 - R_2$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & -2 & 2 & -4 & -2 \\ 0 & 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$R_3 \times (-1/2)$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & -1 & 1 & -2 & -1 \end{array}\right]$$

$R_4 + R_3$

$$\sim \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 3$; $\rho(A|D) = 3$

$\rho(A) = \rho(A|D) <$ No. of unknown(5)

No. of var = No. of " $- \rho(A)$

$= 5 - 3 = 2$

Let $x_5 = k_1$; $x_4 = k_2$

In matrix form,

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x_3 - x_4 + 2x_5 = 1$$

$$x_3 = 1 + k_2 - 2k_1$$

$$x_2 + x_3 + 2x_4 + 2x_5 = 0$$

$$x_2 = -1 - k_2 + \cancel{2k_1} - 2k_2 - \cancel{2k_1}$$

$$x_2 = -1 - 3k_2$$

$$x_1 - x_2 + x_3 - x_4 + x_5 = 1$$

$$x_1 = 1 - 1 - 3k_2 - 1 - \cancel{k_2} + 2k_1 + \cancel{k_2} - k_1$$

$$= k_1 - 3k_2 - 1$$

(vii) In matrix form :-

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 14 \end{bmatrix}$$

$$A \cdot X = D$$

Aug mat is

$$[A \,\vdots\, D] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 7 & 14 \end{array}\right]$$

$4R_1 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 9 & 22 & 28 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 7 & 14 \end{array}\right]$$

$R_2 - 7R_1 \; ; \; R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 9 & 22 & 28 \\ 0 & -60 & -156 & -188 \\ 0 & -15 & 7-44 & 14-56 \end{array}\right]$$

$R_2 \times \left(-\frac{1}{60}\right)$

$$\sim \left[\begin{array}{ccc|c} 1 & 9 & 22 & 28 \\ 0 & 1 & 13/5 & 188/60 \\ 0 & -15 & \lambda-54 & \mu-56 \end{array}\right]$$

$R_3 + 15R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 9 & 22 & 28 \\ 0 & 1 & 13/5 & 188/60 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right]$$

i) For no solution;

$\rho(A) \neq \rho(A|D)$

When $\lambda = 5$ & $\mu \neq 9$

$\rho(A) = 2$ & $\rho(A|D) = 3$

ii) For unique soln

$\rho(A) = \rho(A|D) = $ No. of unknown $= 3$

$\lambda \neq 5$ & μ can have any valu

iii) For infinitely many soln.

$\rho(A) = \rho(A|D) <$ No. of unknown

When $\lambda = 5$ & $\mu = 9$

$\rho(A) = \rho(A \vdots D) = 2$

(viii) In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}$$

$A \quad X = D$

Aug. mat. is

$$[A \vdots D] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \eta \\ 1 & 4 & 10 & \eta^2 \end{array}\right]$$

$R_2 - R_1 ; R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \eta-1 \\ 0 & 3 & 9 & \eta^2-1 \end{array}\right]$$

$R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right]$$

$\rho(A) = 2$

For inf many soln

$\rho(A \vdots D) = \rho(A) = 2$

$\therefore \lambda^2 - 3\lambda + 2 = 0$

$\therefore \lambda = 1, 2$

When $\lambda = 1$

In mat form

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $z = k_1$

$y + 3z = 0$

$y = -3k_1$

$x + y + z = 1$

$x = 1 + 2k_1$

When $n=2$

In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Let $z = k_2$

$y + 3z = 1$

$y = 1 - 3k_2$

$x + y + z = 1$

$x = 2k_2$

TYPE III :- HOMOGENEOUS EQN :-

Homog. eqn. have RHS = 0

i) If $|A| \neq 0$ then eqn. have zero soln. or trivial or unique soln and $\rho(A) = \rho(A|O) =$ No. of unk

ii) If $|A| = 0$ then eqn. have

infinitely many soln or non-zero or non-trivial soln &

$\rho(A) = \rho(A|O) <$ No. of unknowns

i) In matrix form

$$\begin{bmatrix} 4 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \cdot X = O$$

$|A| = 47 \neq 0$

Eqn have zero soln

$x = y = z = 0$

ii) In matrix form

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \cdot X = O$$

$|A| = 0$

Aug. matrix is,

$$[A \mid O] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & -4 & 5 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

$R_2 - 3R_1$; $R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 14 & -14 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right]$$

$R_2 \times (1/14)$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right]$$

$R_3 - 7R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \rho(A|O) = 2 <$ No. of unknowns

No. of var. $= 3 - 2 = 1$

Let $z = k$

In mat. form

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y - z = 0$

$y = k$

$x - 4y + 5z = 0$

$x = -k$

(iii) In matrix form,

$$\begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\because$ $|A|$ is diff. to find beg. with $[A \mid 0]$

Aug. mat. is :-

$$[A \mid 0] = \left[\begin{array}{cccc|c} 2 & -2 & 5 & 3 & 0 \\ 4 & -1 & 1 & 1 & 0 \\ 3 & -2 & 3 & 4 & 0 \\ 1 & -3 & 7 & 6 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_4$

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 7 & 6 & 0 \\ 4 & -1 & 1 & 1 & 0 \\ 3 & -2 & 3 & 4 & 0 \\ 2 & -2 & 5 & 3 & 0 \end{array}\right]$$

$R_2 - 4R_1$; $R_3 - 3R_1$; $R_4 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & -3 & 7 & 6 & 0 \\ 0 & 11 & -27 & -23 & 0 \\ 0 & 7 & -18 & -14 & 0 \\ 0 & 4 & -9 & -9 & 0 \end{array}\right]$$

$(R_2 - 3R_4)$

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 7 & 6 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 7 & -18 & -14 & 0 \\ 0 & 4 & -9 & -9 & 0 \end{array}\right]$$

$R_3 - 7R_2 \; ; \; R_4 - 4R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 7 & 6 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & -18 & 14 & 0 \\ 0 & 0 & -9 & 7 & 0 \end{array}\right]$$

$(2R_4 - R_3)$

$$\sim \left[\begin{array}{cccc|c} 1 & -3 & 7 & 6 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & -18 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 3 \; ; \; \rho(A \vdots D) = 3$

$\rho(A) = \rho(A \vdots D) <$ No. of unknown (4)

No. of var. $= 4 - 3 = 1$

Let $w = k_1$

In mat. form

$$\begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & -18 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-18z + 14w = 0$

$\therefore z = \dfrac{14k_1}{18} = \dfrac{7k_1}{9}$

$y - 4w = 0 \quad \therefore y = 4k_1$

$x - 3y + 7z + 6w = 0$

$x = 12k_1 - \dfrac{49k_1}{9} - 6k_1$

$= \dfrac{5k_1}{9}$

(iv) Considering three eqns.

In mat. form

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \quad X = O$$

$$|A| = -7 \neq 0$$

$$\therefore x_1 = x_2 = x_3 = 0$$

(v) In mat. form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 3 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \cdot X = 0$$

Aug. mat. is

$$[A \mid O] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & -1 & -1 & 0 \\ 3 & -1 & 2 & 3 & 0 \end{array}\right]$$

$R_2 - R_1$; $R_3 - 3R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & -1 & -4 & -2 & 0 \\ 0 & -7 & -7 & 0 & 0 \end{array}\right]$$

$R_3 - 7R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & -1 & -4 & -2 & 0 \\ 0 & 0 & 21 & 14 & 0 \end{array}\right]$$

$\rho(A) = 3$; $\rho(A \vdots D) = 3$

$\rho(A) = \rho(A \vdots D) < \text{No. of unknown}(4)$

No. of var. $= 4 - 3 = 1$

Let $x_4 = k$,

In mat. form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & 21 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$21x_3 + 14x_4 = 0$$

$$x_3 = -\frac{2k_1}{3}$$

$$-x_2 - 4x_3 - 2x_4 = 0$$

$$x_2 = \frac{8k_1}{3} - 2k_1 = \frac{2k_1}{3}$$

$$x_1 + 2x_2 + 3x_3 + x_4 = 0$$

$$x_1 = -\frac{4k_1}{3} + 2k_1 - k_1 = -\frac{k_1}{3}$$

vi) In mat. form

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \quad X = 0$$

For non-zero soln.

$$|A| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[1-2\lambda+\lambda^2-6] - 2(3-3\lambda-4)$$
$$3(9-2+2\lambda) = 0$$

$$(1-\lambda)(\lambda^2-2\lambda-5)-2(-3\lambda-1)+3(2\lambda+7)=0$$

$$\lambda^2-2\lambda-5-\lambda^3+2\lambda^2+5\lambda+6\lambda+2+6\lambda+21=0$$

$$-\lambda^3+3\lambda^2+15\lambda+18=0$$

$$\lambda=6$$

vii) In mat form

$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A \quad X = 0$$

For non-zero soln.

$$|A|=0$$

$$\begin{vmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{vmatrix} = 0$$

$R_1 - 2R_3$; $R_2 - R_3$

$$\begin{vmatrix} 0 & -k-4 & -3k-4 \\ 0 & -k+2 & k-2 \\ 1 & 2k+2 & 3k+4 \end{vmatrix} = 0$$

$(k+4)[2-k] + (3k+4)[2+k] = 0$

$(2-k)(k+4+3k+4) = 0$

$\cancel{(2-k)(-2k) = 0}$

$(2-k)(4k+8) = 0$

$k = \pm 2$

For zero soln.

$|A| \neq 0$

$k \neq \pm 2$

TYPE IV :- LINEAR DEPENDENC & INDEP. OF VECTORS

We find dep. rel. by using method to solve homog eqns.

If $|A| \neq 0$ then $C_1 = C_2 = \ldots = C_n = 0$

$\therefore$ Vector are linearly independent

If $|A| = 0$ then there are ∞ many soln. & vecters are linearly dep.

1) $X_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$; $X_2 = \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix}$

Let $C_1 X_1 + C_2 X_2 = 0$

$$C_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ 7 \\ 10 \end{bmatrix} = 0$$

$c_1 + 3c_2 = 0$

$2c_1 + 7c_2 = 0$

$4c_1 + 10c_2 = 0$

Consid. 2 eqns.

In mat. form

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$A \cdot X = 0$

$|A| = 1 \neq 0$

$\therefore c_1 = c_2 = 0$

$\therefore$ Vectors ~~have~~ are linearly indep.

2)

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; X_3 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \; X_3 = \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$$

Let $c_1 X_1 + c_2 X_2 + c_3 X_3 = 0$

$$c_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 + 3c_2 + c_3 = 0$$
$$2c_1 - 2c_2 - 6c_3 = 0$$
$$3c_1 + c_2 - 5c_3 = 0$$

In mat form

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

$$|A| = 0$$

Aug mat :

$$[A \mid O] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & -2 & -6 & 0 \\ 3 & 1 & -5 & 0 \end{array}\right]$$

$R_2 - 2R_1$; $R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -8 & -8 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right]$$

$R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -8 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 \; ; \; \rho(A|O) = 2$

$\rho(A) = \rho(A|O) <$ No. of unknown(3)

No. of var. $= 3 - 2 = 1$

Let $c_3 = K$

In mat form

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-8c_2 - 8c_3 = 0$

$c_3 = -c_2$

$c_2 = -K$

$c_1 + 3c_2 + c_3 = 0$

$c_1 = 3k - k = 2k$

$\therefore c_1X_1 + c_2X_2 + c_3X_3 = 0$

$2X_1 - X_2 + X_3 = 0$ is linear dep. relation

iii) Let $c_1X_1 + c_2X_2 + c_3X_3 = 0$

$c_1(1\ 2\ 4) + c_2(2\ -1\ 3) + c_3(0\ 1\ 2) = (0\ 0\ 0)$

$c_1 + 2c_2 + 0c_3 = 0$

$2c_1 - c_2 + c_3 = 0$

$4c_1 + 3c_2 + 2c_3 = 0$

In mat. form

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A \cdot X = 0$

$|A| = -5 \neq 0$

$\therefore$ eqns. are linearly indep.

& $c_1 = c_2 = c_3 = 0$

(v)

Let $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0$

$c_1 + 2c_2 + 0c_3 - 3c_4 = 0$

$2c_1 - c_2 + c_3 + 7c_4 = 0$

$4c_1 + 3c_2 + 2c_3 + 2c_4 = 0$

In mat. form

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~$C_2 - 2C_1 \; ; \; C_3 - 4C_1$~~

Aug. mat. is

$$[A \,|\, 0] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 4 & 3 & 2 & 2 & 0 \end{array}\right]$$

$C_2 - 2C_1 \; ; \; C_3 - 4C_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 0 \\ 0 & -5 & 1 & 13 & 0 \\ 0 & -5 & 2 & 14 & 0 \end{array}\right]$$

$C_3 - C_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -3 & 0 \\ 0 & -5 & 1 & 13 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$\rho(A) = 3$; $\rho(A|O) = 3$

$\rho(A) = \rho(A|O) <$ No of unknown/4

No. of var. $= 4-3 = 1$

Let $c_4 = k_1$

In mat. form

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_3 + c_4 = 0$$

$$c_3 = -k_1$$

$$-5c_2 + c_3 + 13c_4 = 0$$

$$c_2 = \frac{-k_1 + 13k_1}{5} = \frac{12k_1}{5}$$

$$c_1 + 2c_2 - 3c_3 = 0$$

$$c_1 = -3k_1 - \frac{24k_1}{5} = -\frac{39k_1}{5}$$

$\therefore c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0$

$-3\cancel{9}X_1 + 12X_2 - 5X_3 + 5X_4 = 0$

TYPE V :- SPECIAL MATRICES:-

1) Idempotent mat. - Mat. A is
" if $A^2 = A$

eg. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

2) Involutory - Mat. A is
" if $A^2 = I$ where
I is a unit mat.

eg $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

3) Nilpotent mat - Mat A is
'' if $A^m = 0$ where
0 is a null mat & m
is the index

eg. for $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$A^3 = 0$ $\therefore$ Index = 3

4) Conjugate of mat. - is obtained by replacing each element of the mat. by its conjugate

eg. If $A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$

5) Mat. A^{θ} is conj. of $\bar{A}$

eg

$$A = \begin{bmatrix} 2+3i & 4i & 7 \\ 9+i & 2-7i & i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 2-3i & -4i & 7 \\ 9-i & 2+7i & -i \end{bmatrix}$$

$$A^{\theta} = (\bar{A})' = \begin{bmatrix} 2-3i & 9-i \\ -4i & 2+7i \\ 7 & -i \end{bmatrix}$$

6) Unitary Mat – A square A is " if $A^{\theta} A = I$

eg $A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$

) Hermitian Mat – A square mat A is " if $A = A^{\theta}$

8) Skew-Hermitian - A square mat. A is " if $A = -A^\theta$

eg $A = \begin{bmatrix} i & 2-3i & 4+5i \\ -2-3i & 0 & 2i \\ -4+5i & 2i & -3i \end{bmatrix}$